TANGENT CONE OF NUMERICAL SEMIGROUP RINGS WITH SMALL EMBEDDING DIMENSION

YIHUANG SHEN

Abstract. In this paper, we study the tangent cone of numerical semigroup rings with small embedding dimension d. For $d = 3$, we give characterizations of the Buchsbaum and Cohen-Macaulay properties and for $d = 4$, we give a characterization of the Gorenstein property. In particular, when $d = 4$ and the tangent cone is Gorenstein, the initial form ideal of the defining ideal is 5-generated.

1. Introduction

Throughout this paper we fix $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. Recall that a numerical semigroup $G = \langle n_1, \cdots, n_d\rangle$ generated by $n_1, \dots, n_d \in \mathbb{N}$ is the set $\{\sum_{i=1}^n a_i n_i : a_i \in \mathbb{N}_0\}$. For simplicity, we always assume that G is minimally generated by these generators, with $n_1 < \cdots < n_d$ and $\gcd(n_1, \dots, n_d) = 1$, unless stated otherwise. Let k be a field of characteristic 0, and t an indeterminate over k. Then as a subring of the power series ring $V = k[[t]]$, $R = k[[t^{n_1}, \dots, t^{n_d}]]$ is the numerical semigroup ring associated to G with $\mathfrak{m} = (t^{n_1}, \dots, t^{n_d})R$ being the unique maximal ideal.

R is the homomorphic image of the power series ring $S = k[[x_1, \dots, x_d]]$ by mapping x_i to t^{n_i}. Let $\mathfrak{n}$ be the unique maximal ideal of S, and I the kernel of this surjective map. I is a binomial ideal. We denote the kernel of the natural map $\mathrm{gr}_{\mathfrak{n}}(S) \to \mathrm{gr}_{\mathfrak{m}}(R)$ by I^*, and call it the *initial form ideal* of I.

When the embedding dimension $d = 3$, Herzog [11] gave a complete characterization of the defining ideal I. In particular the minimal number of generators $\mu(I) \le 3$. It is also proven by Robbiano and Valla [14] and Herzog [12] that the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if the initial form ideal I^* is generated by at most 3 elements. In this paper, we give another characterization for the Cohen-Macaulay property in terms of the index of nilpotency $\mathrm{s}_Q(\mathfrak{m})$ and the reduction number $\mathrm{r}_Q(\mathfrak{m})$, where $Q = (t^{n_1})R$ is a principal reduction of $\mathfrak{m}$. Recall that $\mathrm{s}_Q(\mathfrak{m}) = \min\{s \mid \mathfrak{m}^{s+1} \subseteq Q\}$.

For $d = 3$, we also study the 0-th local cohomology module of the tangent cone. And we are able to characterize the k-Buchsbaum properties of $\mathrm{gr}_{\mathfrak{m}}(R)$ for $k = 1, 2$, in terms of the length of this local cohomology module. In particular, this answers the conjectures raised by Sapko [15].

When $d = 4$ and the numerical semigroup G is symmetric, Bresinsky [3] gave a complete description of the defining ideal I. In particular, it is well-known now that $\mu(I) \le 5$. In this paper, we also study the initial form ideal of I. When the tangent cone $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein, we show that I^* is also 5-generated. Meanwhile, Arslan

1991 *Mathematics Subject Classification.* Primary 13A30, Secondary 13P10, 13H10.

[1] and Shibuta [16] showed that $\mu(I^*)$ could be arbitrarily large when $\mathrm{gr}_{\mathfrak{m}}(R)$ is only Cohen-Macaulay.

How about $d = 5$? In this situation, Bresinsky [4] proved that $\mu(I) \leq 13$, under the condition that $n_1 + n_2 = n_3 + n_4$. Here in $G = \langle n_1, n_2, \cdots, n_5\rangle$, he didn't assume the generators to be in increasing order. Furthermore, our computations suggest a positive answer to the following question.

Question 1.1. If the symmetric numerical semigroup G has embedding dimension 5, is $\mu(I) \leq 13$ in general? If the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein, is $\mu(I^*) \leq 13$ as well?

The main technique in this paper is to play with the standard basis of the defining ideal I. The standard basis algorithm can generate a set of standard basis from a minimal generating set of I. The new generators are *s-polynomials*. Recall that for a polynomial ring $A = k[\underline{X}]$ with monomial ordering $<_A$, the s-polynomial for f_1 and f_2 in A is defined to be

$$\mathrm{spoly}(f_1, f_2) = \frac{\mathrm{lcm}(LM(f_1), LM(f_2))}{LM(f_1)} f_1 - \frac{\mathrm{lcm}(LM(f_1), LM(f_2))}{LM(f_2)} f_2.$$

$LM(f)$ is the leading monomial of f. When we have a local monomial ordering, we sometimes also call it the *initial monomial* of f. It is part of the initial form of f. And in this case, the standard basis algorithm is sometimes called the Mora form algorithm. When the Krull dimension of A is small, it is possible in some cases to carry out the standard basis algorithm by hand.

As an application of the above considerations, we are able to answer a question raised by Heinzer and Swanson which is related to [10]. In this paper, they introduced the Goto numbers for parameter ideals. If $(R, \mathfrak{m})$ is a Noetherian local ring and Q is a parameter ideal, then the *Goto number* $g(Q) = \max\left\{i \mid (Q : \mathfrak{m}^i) \subseteq \bar{Q}\right\}$, where $\bar{Q}$ is the integral closure of Q.

For numerical semigroup ring R, Heinzer and Swanson [10, 4.1] proved that

$$g(t^{f+n_1+1}) = \min\left\{g(Q) \mid Q \text{ is a parameter ideal of } R\right\}.$$

where $f = \max\left\{x \in \mathbb{N}_0 \mid x \notin G\right\}$ is the Frobenius number of G.

In particular, one has

$$(\ddagger) \qquad g(t^{f+n_1+1}) \leq \min\left\{g(t^{n_i}) \mid i = 1, \ldots, d\right\}.$$

Equality holds in $(*)$ where $d = 2$ (c.f. [10, 5.10]). It is natural to ask if equality holds for $d \geq 3$. We note that for $d = 3$, the numerical semigroup ring R corresponding to $\langle 11, 14, 21\rangle$ is Gorenstein. But strict inequality holds for $(*)$. Heinzer pointed out that the tangent cone of this example is Cohen-Macaulay, but not Gorenstein. And we will show that if the tangent cone is Gorenstein, then equality holds for $(\ddagger)$ for $d = 3, 4$. A general proof is given by Lance Bryant.

2. Goto numbers

The conductor ideal C of R with respect to V is defined by $C = t^{f+1}V$ where f is the Frobenius number.

Let e be an nonzero element in G. Recall that the *Apéry set* of G with respect to e is $\mathrm{Ap}(G, e) = \{a(0), \cdots, a(e-1)\}$, where $a(i)$ is the smallest element in G congruent with i modulo e. Usually we write the elements of $\mathrm{Ap}(G, e)$ in increasing

order: $\omega_0 = 0 < \omega_1 < \cdots < \omega_{e-1} = e + f$. When $e = n_1$, the multiplicity, the following conditions are equivalent.

(a) The numerical semigroup ring R is Gorenstein.
(b) The numerical semigroup G is symmetric in the sense that for every $z \in \mathbb{Z}$, $z \in G$ if and only if $f - z \notin G$.
(c) $\omega_i + \omega_{e-1-i} = \omega_{e-1}$ where $0 \leq i \leq e - 1$.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a Gorenstein numerical semigroup ring, and $Q = t^{n_1}R$. Then the index of nilpotency $\mathrm{s}_Q(\mathfrak{m}) = \mathrm{ord}_{\mathfrak{m}}(t^{f_1+n_1})$.*

Proof. We always have $\mathrm{s}_Q(\mathfrak{m}) = \max\{\mathrm{ord}(w_i) : 1 \leq i \leq n_1 - 1\}$, and when G is symmetric, $\mathrm{ord}_{\mathfrak{m}}(w_{n_1-1})$ is the largest. Hence $\mathrm{s}_Q(\mathfrak{m}) = \mathrm{ord}_{\mathfrak{m}}(t^{f+n_1})$. □

For every $z \in G$, it is obvious that $\mathrm{ord}_{\mathfrak{m}}(t^z) = \max\{\sum a_i : \sum a_i n_i = z, a_i \in \mathbb{N}_0\}$. When there is on confusion, we also write this number as $\mathrm{ord}_G(z)$ and similarly define $\text{min-ord}_G(z)$ to be $\min\{\sum a_i : \sum a_i n_i = z, a_i \in \mathbb{N}_0\}$. The ratio $\frac{\mathrm{ord}_G(z)}{\text{min-ord}_G(z)}$ is called the *elasticity of z with respect to G.*

Lemma 2.2. *If G is symmetric, then* $\text{min-ord}_G(f + n_1) \leq \mathrm{ord}_{\mathfrak{m}}(C)$.

Proof. Set $Q = t^{n_1}R$. Any $z \in G$ can be written uniquely in the form $z = an_1 + w_i$ where $a = \mathrm{ord}_Q(z)$ and $w_i \in \mathrm{Ap}(G, n_1)$. If $z > f$, then $an_1 + w_i > w_i + w_{s-1-i} - n_1$. Thus $(a + 1)n_1 > w_{s-1-i}$. If we write $w_{s-1-i} = \sum_j a_j n_j$ with $a_j \in \mathbb{N}_0$, then $a_1 = 0$, and $w_{s-1-i} > \sum a_j n_1$. Consequently $a + 1 > \sum_j a_j$, i.e., $a \geq \sum_j a_j$. Hence $a \geq \mathrm{ord}_G(w_{s-1-i})$. Now $\mathrm{ord}_{\mathfrak{m}}(z) \geq a + \mathrm{ord}_G(w_i) \geq \mathrm{ord}_G(w_{s-1-i}) + \mathrm{ord}_G(w_i) \geq \text{min-ord}_G(f + n_1)$. Since the elements of the form t^z with $z > f$ generate the ideal C, this completes the proof. □

Example 2.3. In general, it is not true that $\mathrm{ord}_G(f+n_1) \leq \mathrm{ord}_{\mathfrak{m}}(C)$. For example, if $G = \langle 11, 14, 21\rangle$, then G is symmetric and the Frobenius number $f = 73$. It is not difficult to see that $\mathrm{ord}_G(f + n_1) = 6$ while $\mathrm{ord}_{\mathfrak{m}}(C) = 5$.

Inspired by the proof of 2.2, we define a partial order $\preceq_G$ on elements in G. We say $x \preceq_G x'$ for $x, x' \in G$ if we can write $x = \sum_i a_i n_i$ and $x' = \sum_i a'_i n_i$ with $a_i, a'_i \in \mathbb{N}_0$, such that $\sum a_i = \mathrm{ord}_G(x)$, $\sum_i a'_i = \mathrm{ord}_G(x')$, and $a_i \leq a'_i$ for all i. Roughly speaking, $x \preceq_G x'$ if and only if "x has a maximal decomposition with respect to G that is dominated by a maximal decomposition of x' with respect to G".

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a Gorenstein numerical semigroup ring, where the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. If for every $x \in \mathrm{Ap}(G, n_1)$, $x \preceq_G (f + n_1)$, then equality holds for (‡) in the introduction. In addition,*

$$g(t^{f+n_1+1}) = g(t^{n_1}) = \mathrm{r}_Q(\mathfrak{m}) = \mathrm{ord}_{\mathfrak{m}}(C) = \mathrm{ord}_G(f + n_1).$$

Proof. According to the proof of 2.2, for any element $z \in G$ with $z > f$, $z - an_1 = w_j \in \mathrm{Ap}(G, n_1)$ with $a = \mathrm{ord}_Q(z)$. Since $w_j \preceq_G (f+n_1)$, we can write $w_j = \sum a_i n_i$, $f + n_1 = \sum b_i n_i$, with $a_i, b_i \in \mathbb{N}_0$, $a_i \leq b_i$ for $2 \leq i \leq d$. Now write $a'_i = b_i - a_i$ and one has $w_{s-1-j} = f + n_1 - w_i = \sum a'_i n_i$. Hence by the proof of 2.2,

$$\mathrm{ord}_G(z) \geq \mathrm{ord}_G(w_j) + \mathrm{ord}_G(w_{s-1-j}) \geq \sum b_i = \mathrm{ord}_G(f + n_1).$$

This implies that $\mathrm{ord}_{\mathfrak{m}}(C) \geq \mathrm{ord}_G(f+n_1)$. By virtue of 2.1, we have $\mathrm{ord}_G(f+n_1) \geq \mathrm{r}_Q(\mathfrak{m}) = g(t^{n_1})$. On the other hand, Theorem 4.1 and Corollary 5.8 of [10] yield

$g(t^{n_1}) \geq g(t^{f+n_1+1}) \geq \mathrm{ord}_{\mathfrak{m}}(C)$. Since

$$\mathrm{ord}_{\mathfrak{m}}(C) = \min\left\{\mathrm{ord}_G(z) \mid z \in G, z > f\right\},$$

$\mathrm{ord}_{\mathfrak{m}}(C) \geq \mathrm{ord}_G(f+n_1)$. Now the equalities in the assertion follow immediately. In particular, equality holds for (‡). □

Remark 2.5. If $\mathrm{ord}_G(f+n_1) = \text{min-ord}_G(f+n_1)$, i.e., the elasticity of $f+n_1$ with respect to G is 1, then every decomposition of $f+n_1$ is maximal. Hence for any $x \in \mathrm{Ap}(G, n_1)$, $x \preceq_G (f+n_1)$. This elasticity condition is satisfied when the embedding dimension equals 4, the tangent cone $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein but R is not complete intersection, as we demonstrate in 5.16.

Corollary 2.6. *For numerical semigroup ring* $(R, \mathfrak{m})$ *with embedding dimension* $d \leq 4$, *if the associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein, then*

$$g(t^{f+n_1+1}) = g(t^{n_1}) = \mathrm{r}(\mathfrak{m}) = \mathrm{ord}_{\mathfrak{m}}(C) = \mathrm{ord}_G(f+n_1).$$

In particular, equality holds for (‡).

When $d = 4$, it follows from 2.4 and 5.1. The case when $d = 3$ will be an easy exercise.

It is pointed out by Lance Bryant that the partial order condition on the Apéry set in 2.4 is equivalent to the following condition for multiplicity $e = n_1$:

(†) $\forall w_i, w_j \in \mathrm{Ap}(G, n_1), w_i + w_j = w_{e-1} \Rightarrow \mathrm{ord}_{\mathfrak{m}}(w_i) + \mathrm{ord}_{\mathfrak{m}}(w_j) = \mathrm{ord}_{\mathfrak{m}}(w_{e-1})$.

And he gives the following characterization

Theorem 2.7 ([6, 3.20])**.** *Suppose G is a symmetric numerical semigroup, and* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay. Then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein if and only if the condition* (†) *holds. In particular, Corollary 2.6 holds with no restriction on the embedding dimension.*

3. Initial form ideals

Following our previous notation, $S = K[[x_1, \dots, x_d]]$ with maximal ideal $\mathfrak{n}$ maps onto R. For each nonzero $x \in S$, let $o = \mathrm{ord}_{\mathfrak{n}}(x) < \infty$ be the $\mathfrak{n}$-adic order of x. We denote by x^* the residue class of x in $\mathfrak{n}^o/\mathfrak{n}^{o+1}$ and call it the initial form of x. The initial form ideal $I^* \subset \mathrm{gr}_{\mathfrak{n}}(S)$ is generated by x^* for all $x \in I$. For our numerical semigroup ring R, the radical of the initial ideal I^* is very simple.

Lemma 3.1. $\sqrt{I^*} = (x_2^*, \dots, x_d^*)\,\mathrm{gr}_{\mathfrak{n}}(S)$.

Proof. For all i with $2 \leq i \leq d$, $f_i = x_i^{n_1} - x_1^{n_i} \in I$. Since $n_1 < n_i$, the initial form of f_i^* is $(x_i^*)^{n_1} \in I^*$. Hence $(x_2^*, \dots, x_d^*) \subseteq \sqrt{I^*}$. Since $\mathrm{ht}(x_2^*, \dots, x_d^*) = \mathrm{ht}\, I^* = \mathrm{ht}\, I = d-1$ and $(x_2^*, \dots, x_d^*)$ is a prime ideal in $\mathrm{gr}_{\mathfrak{n}}(S)$, $\sqrt{I^*} = (x_2^*, \dots, x_d^*)$. □

Since $\mathrm{gr}_{\mathfrak{n}}(S) \cong k[x_1, \dots, x_d]$, by abuse of notation, in the sequel of this paper we simply write x_i^* as x_i for $1 \leq i \leq d$, when there is no confusion.

We need to go over some basic definitions and notations of standard basis. Given a fixed local monomial order ring $>$, for a nonzero $f \in S$, there is a unique way to write

$$f = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_t x^{\alpha_t}, \qquad x^{\alpha_1} > x^{\alpha_2} > \cdots > x^{\alpha_t}.$$

$\mathrm{LM}(f) := x^{\alpha_1}$ is called the *initial monomial* of f. The *initial ideal* $\mathrm{LI}(I)$ of I is generated by $\mathrm{LM}(f)$ for $f \in I$. A subset $F := \{f_1, \dots, f_s\} \subset I$ is a *standard basis*

of I if $\mathrm{LI}(I) = (\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_s))$. F is called *minimal* if $\mathrm{LM}(f_i) \nmid \mathrm{LM}(f_j)$ for $i \neq j$.

A binomial $f = x^\alpha - x^\beta$ with $\alpha = (\alpha(1), \ldots, \alpha(d)), \beta = (\beta(1), \ldots, \beta(d)) \in \mathbb{N}_0^d$ is called *weakly balanced* if $\sum_i \alpha(i) n_i = \sum_i \beta(i) n_i$. f is called *balanced* if it is weakly balanced, $\deg x^\alpha = x^\beta$, and x^α and x^β form a regular sequence.

For the numerical semigroup ring R, the defining ideal I is generated by weakly balanced binomials. Applying the standard basis algorithm with suitable monomial ordering to this generating set, one is able to get a minimal standard basis $\{f_1, \ldots, f_s\}$. And I^* is minimally generated by the corresponding initial forms: $I^* = (f_1^*, \ldots, f_s^*)$. Since each f_i is also a weakly balanced binomial, f_i^* is either a monomial or a balanced binomial. In the latter case, roughly speaking, $f_i = f_i^*$. In the rest of the paper, when we say g is a minimal generator of I^*, it is understood that $g \in \{f_1^*, \ldots, f_s^*\}$ when a minimal generating set of I and the monomial ordering is clear.

From now on, we need to choose the monomial ordering $>$ more carefully. A monomial ordering $>$ is *nice* in variable x_i if the following holds:

$$\deg x^\alpha < \deg x^\beta, \text{ or } (x^\alpha - x^\beta \text{ is balanced}, \beta(i) > 0) \implies x^\alpha > x^\beta.$$

Being nice is really a mild condition. For instance, the modified negative degree reverse lexicographical ordering in the following is nice in x_1:

$$\begin{aligned} x^\alpha > x^\beta :\Longleftrightarrow \deg x^\alpha < \deg x^\beta, \text{ or } (&\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \\ &\alpha(1) = \beta(1), \ldots, \alpha(i-1) = \beta(i-1), \alpha(i) < \beta(i)). \end{aligned}$$

When $d = 3$, the normal negative degree reverse lexicographical ordering is also nice in x_1.

An ideal $J \subset T := k[x_1, \ldots, x_d]$ is called *almost balanced* if it satisfies the following conditions:

(a) $\sqrt{J} = (x_2, \ldots, x_d)$;
(b) there is a minimal standard basis $\{f_1, \ldots, f_t\}$ of J, such that f_i is either a monomial or a balanced binomial.

The initial form ideal I^* is almost balanced.

Lemma 3.2. *Let $>$ be a nice monomial ordering in x_1 for T and J an almost T-ideal. Then T/J is Cohen-Macaulay if and only if x_1 does not divide any of the monomial generators of J constructed above.*

Proof. T/J is Cohen-Macaulay if and only if x_1 is a regular element.

If in the previous definition that some $f_i = x_1 x^\alpha$, then $x^\alpha \notin J$, since $\{f_1, \ldots, f_s\}$ is a minimal standard basis. Hence J is not a perfect ideal.

On the other hand, suppose that $x_1 f \in J$ and $0 \neq f \notin J$. We may replace f by the normal form $\mathrm{NF}(f | \{f_1, \ldots, f_s\})$ to assume that none of the monomials in f is divisible by any of $\mathrm{LM}(f_i)$. Notice that $x_1 \mathrm{LM}(f) = \mathrm{LM}(x_1 f)$ is divisible by some $\mathrm{LM}(f_i)$. But $\mathrm{LM}(f)$ is not divisible by $\mathrm{LM}(f_i)$, hence $\mathrm{LM}(f_i)$ is divisible by x_1. Since the ordering is nice in x_1, f_i cannot be a balanced binomial, hence it is a monomial. $\square$

Example 3.3. Let G be the numerical semigroup generated by 5, 6 and 13. Then the defining ideal $I = (x_2^2 x_3 - x_1^5, x_3^2 - x_1^4 x_2, x_1 x_3 - x_2^3)$. With the normal negative degree reverse lexicographical ordering, $\left\{x_2^2 x_3 - x_1^5, x_3^2 - x_1^4 x_2, x_1 x_3 - x_2^3, x_2^5 - x_1^6\right\}$ form a minimal standard basis. Thus the initial form ideal is $I^* = (x_2^5, x_2^2 x_3, x_3^2, x_1 x_3)$.

Since x_1x_3 is divisible by x_1, I^* is not a Cohen-Macaulay ideal. This property also follows from the fact that I^* is generated by more than 3 elements.

For the rest of this paper, we fix

$$\alpha_i = \min\{\alpha \in \mathbb{N} \mid \alpha n_i \in \langle n_1, \dots, \widehat{n_i}, \dots, n_d\rangle\}$$

for $1 \le i \le d$. In the previous example, $\alpha_1 = 5$, $\alpha_2 = 3$ and $\alpha_3 = 2$.

4. When the embedding dimension $d = 3$

Throughout this section, we use negative degree reverse lexicographical ordering on $\operatorname{gr}_{\mathfrak{n}}(S) \cong k[x_1, x_2, x_3]$. Hence if $f = x_2^b - x_1^a x_3^c$ with $a, b, c \in \mathbb{N}$, and $b = a + c$, then the initial monomial of f is x_2^b.

We need the following result on the structure of 3-generated numerical semigroups by Herzog.

Theorem 4.1 ([11, 2.1, 3.8]). *Let G be a numerical semigroup minimally generated by 3 elements.*

(a) *If G is symmetric, then by a permutation (i, j, k) of $(1, 2, 3)$, the defining ideal is*
$$I = (x_i^{\alpha_i} - x_j^{\alpha_j}, x_k^{\alpha_k} - x_i^{r_{ki}} x_j^{r_{ki}}).$$
And the Frobenius number of G is
$$f = (\alpha_i - 1)n_i + (\alpha_k - 1)n_k - n_j.$$

(b) *If G is not symmetric, then*
$$I = (x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}),$$
where all α's are in $\mathbb{N}$ and $\alpha_i = \alpha_{ji} + \alpha_{ki}$ for all permutation (i, j, k) of $(1, 2, 3)$.

In terms of the defining ideal given by 4.1, one can quickly give arithmetic conditions on when $\operatorname{gr}_{\mathfrak{m}}(R)$ will be Cohen-Macaulay.

Corollary 4.2. (a) *If $I = (x_1^{\alpha_1} - x_2^{\alpha_2}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}})$, then $\operatorname{gr}_{\mathfrak{m}}(R)$ is a complete intersection and I^* is generated by $\{x_2^{\alpha_2}, x_3^{\alpha_3}\}$.*

(b) *If $I = (x_1^{\alpha_1} - x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}})$, then $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if $\alpha_2 \le \alpha_{21} + \alpha_{23}$. Whence, I^* is generated by*
$$\{x_3^{\alpha_3}, x_2^{\alpha_2} \text{ or } x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}\}.$$

(c) *If $I = (f_1 := x_2^{\alpha_2} - x_3^{\alpha_3}, f_2 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}})$ with $\alpha_{13} < \alpha_3$. Set $f_3 :=$ $\operatorname{spoly}(f_1, f_2) = x_2^{\alpha_2 + \alpha_{12}} - x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_{13}}$. Then $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if $\alpha_2 + \alpha_{12} \le \alpha_1 + \alpha_3 - \alpha_{13}$. Whence, I^* is generated by $\{x_3^{\alpha_3}, x_2^{\alpha_{12}} x_3^{\alpha_{13}}, f_3^*\}$.*

(d) *If $I = (x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}})$, then $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if $\alpha_2 \le \alpha_{21} + \alpha_{23}$.*

Recall that for a one-dimensional graded ring A with a unique homogeneous maximal ideal $\mathcal{M}$, A is called a *k-Buchsbaum ring* if $\mathcal{M}^k \cdot H^0_{\mathcal{M}}(A) = 0$. 1-Buchsbaum rings are simply called *Buchsbaum*. And 0-Buchsbaum rings are precisely the Cohen-Macaulay rings. In the following, denote the homogeneous maximal ideal of $\operatorname{gr}_{\mathfrak{n}}(S)$ by $\mathcal{M}$. Since $\operatorname{gr}_{\mathfrak{m}}(R) = \operatorname{gr}_{\mathfrak{n}}(S)/I^*$, we write the image of $f \in \operatorname{gr}_{\mathfrak{n}}(S)$ in $\operatorname{gr}_{\mathfrak{m}}(R)$ as $\overline{f}$.

Sapko [15] investigated the tangent cone of ring R associated with the 3-generated numerical semigroup ring G, and made the following conjectures when the tangent cone is Buchsbaum.

Conjecture 4.3 ([15, Conjecture 24])**.** *If* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Buchsbaum, then for some* $k \geq 1$,

$$0 :_{\operatorname{gr}_m(R)} \mathcal{M} = (\bar{x}_3^k) \operatorname{gr}_{\mathfrak{m}}(R).$$

Conjecture 4.4 ([15, 25])**.** *If* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Buchsbaum, then the initial form ideal* I^* *of* I *is generated by 4 elements.*

Conjecture 4.5 ([15, Conjecture 33])**.** $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Buchsbaum if and only if the length* $\ell(H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R))) \leq 1$.

In this note, we want to give positive answers to the above conjectures when the tangent cone is Buchsbaum, and show similar results when the tangent cone is 2-Buchsbaum.

Lemma 4.6. *Suppose* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is not Cohen-Macaulay. Then the 0-th local cohomology module* $H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R))$ *is principal, and generated by* x_3^γ *for suitable* $\gamma \in \mathbb{N}$.

Proof. The initial form ideal I^* is generated by forms of the following 4 types:
(a) $x_3^{\alpha_3}$,
(b) $x_2^{\gamma_2}$ or balanced $x_2^{\gamma_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}}$,
(c) $x_1^a x_3^c$,
(d) $x_2^b x_3^c$.

There is exactly one generator of the type (a). The same is true for generators of type (b). To see this, notice that if $x_2^{\gamma_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}}$ is balanced, then $x_2^{\gamma_2}$ is its initial monomial. There might be more than one generators of type (c) and (d).

It follows from Corollary 3.2 that I^* is Cohen-Macaulay if and only if generators of type (c) do not exist. If $\operatorname{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay, then $H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R)) \neq 0$. We claim that this local cohomology module is generated by $\bar{x}_3^\gamma$ where

$$\gamma = \min \{c \mid x_1^a x_3^c \text{ is a generator of } I^* \text{ of type (c) for some } a, c \in \mathbb{N}\}.$$

Since $\sqrt{I^*} = (x_2, x_3)$, this $\bar{x}_3^\gamma \in H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R))$. On the other hand, $I^* + (x_3^\gamma)$ is (not necessarily minimally) generated by x_3^γ together with the remaining generators of I^* of type (b) and (d). This last ideal is Cohen-Macaulay by 3.2. □

Corollary 4.7. $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Buchsbaum if and only if* $\ell(H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R))) \leq 1$.

Proposition 4.8. *If* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Buchsbaum, then the initial form ideal* I^* *is 4-generated.*

Proof. When $\operatorname{gr}_{\mathfrak{m}}(R)$ is Buchsbaum and not Cohen-Macaulay, I^* has at least one minimal generator of the form $x_1^{\gamma_1} x_3^{\gamma_3}$. $\bar{x}_3^{\gamma_3} \in H^0_{\mathcal{M}}(\operatorname{gr}_{\mathfrak{m}}(R))$, hence in $\operatorname{gr}_{\mathfrak{n}}(S)$, $\mathcal{M} x_3^{\gamma_3} \subseteq I^*$. Since $x_1^{\gamma_1} x_3^{\gamma_3}$ is a minimal generator, one must have $\gamma_1 = 1$. One also observe that $x_2 x_3^{\gamma_3}, x_3^{\gamma_3+1} \in I^*$. By the minimality of α_3, one has $\alpha_3 = \gamma_3 + 1$. It is also clear now that the only minimal monomial generator that involves positive exponent in x_1 is exactly $x_1 x_3^{\alpha_3 - 1}$. Since $x_2 x_3^{\alpha_3 - 1} \in I^*$, there exists a generator $f = x_2 x_3^\gamma - x_1^\alpha$ of the minimal standard basis with $\gamma \leq \alpha_3 - 1$. However, since $\gamma < \alpha_3$, this f should not be new basis generated from the standard basis algorithm. It should belong to one of the minimal binomial generators of I.

Claim. If R is Gorenstein, then $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if it is Buchsbaum.

By 4.1, when R is Gorenstein, the defining ideal, by a permutation (i,j,k) of $(1,2,3)$, is

$$I = (x_i^{\alpha_i} - x_j^{\alpha_j}, x_k^{\alpha_k} - x_i^{\alpha_{ki}} x_j^{\alpha_{ki}}).$$

By symmetry, we can always assume that $i < j$. Now one can characterize when the associated graded ring is Buchsbaum in terms of these α's. By our discussions for $x_2 x_3^{\alpha_3 - 1}$, the claim is set by checking the minimal binomial generators for the cases when $(i,j,k) = (1,2,3)$ or $(1,3,2)$.

If $(i,j,k) = (2,3,1)$, then

$$I = (f_1 := x_3^{\alpha_3} - x_2^{\alpha_2}, f_2 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}).$$

$f_1^* = x_3^{\alpha_3}$ and we can assume that $0 \le \alpha_{13} < \alpha_3$, hence $\alpha_{12} > 0$. Now $f_2^* = -x_2^{\alpha_{12}} x_3^{\alpha_{13}}$ and it is non-comparable with f_1^*. Apply the standard basis algorithm, we get $f_3 := \mathrm{spoly}(f_1, f_2) = x_2^{\alpha_2 + \alpha_{12}} - x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_{13}}$. Now a necessary and sufficient condition for I^* to be Cohen-Macaulay is $\alpha_2 + \alpha_{12} \le \alpha_1 + \alpha_3 - \alpha_{13}$. Hence, if I^* is Buchsbaum and not Cohen-Macaulay, then $f_3^* = -x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_{13}}$. Notice that f_3 must belong to the minimal standard basis. Now by our discussion above, $\alpha_{12} = \alpha_{12} = 1$ and $\alpha_3 - \alpha_{13} = \alpha_3 - 1$. But if G is minimally generated by 3 elements, then $\alpha_1 > 2$, and this is a contradiction. Thus, the claim if proved.

If follows immediately that if $\mathrm{gr}_{\mathfrak{m}}(R)$ is Buchsbaum, not Cohen-Macaulay, then R is not Gorenstein. Now we can write the defining ideal

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}).$$

Obviously, $f_1^* = -x_2^{\alpha_{12}} x_3^{\alpha_{13}}$ and $f_3^* = x_3^{\alpha_3}$. We can assume that $\alpha_{13}, \alpha_{23} < \alpha_3$. Now by 4.2(d) and our discussion in the beginning, if $\mathrm{gr}_{\mathfrak{m}}(R)$ is Buchsbaum and not Cohen-Macaulay, then $\alpha_2 > \alpha_{21} + \alpha_{23}$, $\alpha_{12} = \alpha_{21} = 1$ and $\alpha_{23} = \alpha_3 - 1$. Now by Theorem 4.1, $\alpha_{13} = 1$, $\alpha_{31} = \alpha_1 - 1$ and $\alpha_{32} = \alpha_2 - 1$. Obviously for $f_4 := \mathrm{spoly}(f_1, f_2) = x_1^{\alpha_1 + 1} x_3^{\alpha_3 - 2} - x_2^{\alpha_2 + 1}$. The Buchsbaumness would require that $\alpha_2 + 1 \le (\alpha_1 + 1) + (\alpha_3 - 2)$. An easy calculation would show that the standard basis algorithm will stop at this step, and $\{f_1, \cdots, f_4\}$ form a minimal standard basis for I. Now

$$I^* = (x_2 x_3, x_1 x_3^{\alpha_3 - 1}, x_3^{\alpha_3}, x_2^{\alpha_2 + 1} \text{ or } x_2^{\alpha_2 + 1} - x_1^{\alpha_1 + 1} x_3^{\alpha_3 - 2}).$$

$\square$

Next, we study the 2-Buchsbaumness of the tangent cone, and still assume that the numerical semigroup G is minimally 3-generated. We want to show the following:

Proposition 4.9. $\mathrm{gr}_{\mathfrak{m}}(R)$ *is 2-Buchsbaum if and only if the length*

$$\ell(H^0_{\mathcal{M}}(\mathrm{gr}_{\mathfrak{m}}(R))) \le 2.$$

Proof. Notice that we assume that the monomial ordering is nice in x_1. Hence if a monomial involving x_1 is the initial monomial of a minimal generator constructed as before, this generator as to be a monomial.

Now if the length is at most 2, the tangent cone is trivially 2-Buchsbaum. On the other hand, we assume that $\mathrm{gr}_{\mathfrak{m}}(R)$ is 2-Buchsbaum. Without loss of generality, we may assume that $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay. Hence in I^*, there is a monomial

minimal generator $x_1^a x_3^c$. Since $x_1^2x_3^c, x_3^2x_3^c \in I^*$, one necessarily have $1 \le a \le 2$ and $\alpha_3 - 2 \le c \le \alpha_3 - 1$.

We claim that there is exactly one such minimal generator with the initial monomial $x_1^a x_3^c$. It is easy to see that this could fail only when both $x_1x_3^{\alpha_3-1}$ and $x_1^2x_3^{\alpha_3-2}$ are minimal generators of I^*. Since they are minimal, there exist $\beta_1, \beta_2 \in \mathbb{N}$ such that both $x_2^{\beta_1} - x_1x_3^{\alpha_3-1}$ and $x_2^{\beta_2} - x_1^2x_3^{\alpha_3-2}$ are both in I. Since $n_3 > n_2 > n_1$, we must have $\beta_1 > \beta_2$. Hence $x_2^{\beta_1-\beta_2}x_1 = x_3$, and G is at most 2-generated, which contradicts our assumption.

Similarly, $x_2^2x_3^c \in I^*$. Hence either it is divisible by the initial monomial $x_2^b x_3^{c'}$ of a minimal generator of I^* with $1 \le b \le 2$ and $1 \le c' \le c$, or $\alpha_2 = 2$. If $\alpha_2 = 2$, then 4.2 implies that I^* is Cohen-Macaulay, and there is no need for further discussion. If $\alpha_2 > 2$, then by an argument similar to that in the previous paragraph, there is exactly one such minimal generator in I^* with initial monomial of the form $x_2^b x_3^{c'}$ with $1 \le b \le 2$ and $1 \le c' \le c$. And this generator must be a monomial.

Now we prove that $\ell(H^0_{\mathcal{M}}(\mathrm{gr}_{\mathfrak{m}}(R))) \le 2$.

(a) Suppose that $x_1^a x_3^c = x_1x_3^{\alpha_3-2}$. Notice that $x_2^2x_3^{\alpha_3-2}, x_2x_3^{\alpha_3-1} \in I^*$. Each of them has to be divisible by some monomial minimal generator of I^* in the form $x_2^b x_3^{c'}$ with $1 \le b \le 2$, $1 \le c' \le c$. But there are at most one such generator. Hence this generator must divide the $\gcd(x_2^2x_3^{\alpha_3-2}, x_2x_3^{\alpha_3-1}) = x_2x_3^{\alpha_3-2}$. In particular, $x_2x_3^{\alpha_3-2} \in I^*$. Hence the vector space $H^0_{\mathcal{M}}(\mathrm{gr}_{\mathfrak{m}}(R)) = (x_3^{\alpha_3-1})\,\mathrm{gr}_{\mathfrak{m}}(R)$ is generated by $\{x_3^{\alpha_3-2}, x_3^{\alpha_3-1}\}$.

(b) The case $x_1^a x_3^c = x_1^2x_3^{\alpha_3-2}$ can never happen. Notice that the image of $x_3^{\alpha_3-2}$ generates the local cohomology module. Hence $x_1x_3 \cdot x_3^{\alpha_3-2} \in I^*$. We know that there are no two such minimal generators of the form $x_1^\alpha x_3^\gamma$ in I^*. Since $x_1^2x_3^{\alpha_3-2}$ is assumed to be a minimal generator, hence $x_1^2x_3^{\alpha_3-2}$ has to divide $x_1x_3^{\alpha_3-1}$, which is impossible.

(c) Assume that $x_1^a x_3^c = x_1x_3^{\alpha_3-1}$. Notice that $x_3^{\alpha_3} \in I^*$. Hence the local cohomology module is generated as vector space by $\{x_3^{\alpha_3-1}\}$ or $\{x_3^{\alpha_3-1}, x_2x_3^{\alpha_3-1}\}$.

(d) Assume that $x_1^a x_3^c = x_1^2x_3^{\alpha_3-1}$. Notice $x_1x_2x_3^{\alpha_3-1} \in I^*$ by the 2-Buchsbaumness. But it can never be a minimal generator. Hence either $x_1x_3^{\alpha_3-1} \in I^*$ or $x_2x_3^{\alpha_3-2} \in I^*$. Because $x_1^2x_3^{\alpha_3-1}$ is a minimal generator, the first option cannot happen. Hence $x_2x_3^{\alpha_3-1} \in I^*$ and the local cohomology module as a vector space is generated by $\{x_3^{\alpha_3-1}, x_1x_3^{\alpha_3-1}\}$.

□

Proposition 4.10. *If* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is 2-Buchsbaum, the initial form ideal* I^* *is 4-generated.*

Proof. We may assume from the beginning that $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay. In particular, $\alpha_2 \ge 3$. If $\mathrm{gr}_{\mathfrak{m}}(R)$ is 2-Buchsbaum, then R is not Gorenstein, by an argument similar to that used in the proof for 4.8. Hence we can assume that R is not Gorenstein as well.

(a) Suppose that $x_1x_3^{\alpha_3-2}$ is one minimal generator for I^*, then

$$I = (f_1 := x_1^{\alpha_1} - x_2x_3^2, f_2 := x_2^{\alpha_2} - x_1x_3^{\alpha_3-2}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_1-1}x_2^{\alpha_2-1})$$

by case (a) of the proof for 4.9 together with 3.2 of [11]. $\mathrm{spoly}(f_1, f_3)$ and (f_1, f_2) will not contribute to the standard basis. Since $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay, $\alpha_2 > 1 + (\alpha_3 - 2)$.

$\alpha_3 - 2$ is an exponent in f_2, thus $\alpha_3 - 2 \geq 1$. If $\alpha_3 = 3$, then x_3 generates the local cohomology module. Hence $x_2^2 x_3 \in I^*$. And there is a standard basis generator $f = x_2^\beta x_3 - x_1^\gamma \in I$ with $\beta = 1$ or 2. But then $f_4 := \mathrm{spoly}(f_1, f_2) = x_2^{\alpha_2+1}x_3 - x_1^{\alpha_1+1}$. Since $n_2 > n_1$ and $\alpha_2 \geq 3 > \beta$, this will imply that $\gamma < \alpha_1$, which is a contradiction.

Hence $\alpha_3 \geq 4$ and $f_4 := \mathrm{spoly}(f_2, f_1) = x_2^{\alpha_2+1} - x_1^{\alpha_1+1}x_3^{\alpha_3-4}$. By the 2-Buchsbaumness, $x_2^{\alpha_2+1}$ has to be the initial monomial. And the standard basis algorithm will stop at this step.

(b) Suppose that $x_1 x_3^{\alpha_3-1}$ is one minimal generator for I^*. Then

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3, f_2 := x_2^{\alpha_2} - x_1x_3^{\alpha_3-1}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_1-1}x_2^{\alpha_2-\alpha_{12}}),$$

with $\alpha_{12} = 1$ or 2. The standard basis algorithm generates $f_4 := \mathrm{spoly}(f_2, f_1) = x_2^{\alpha_2+\alpha_{12}} - x_1^{\alpha_1+1}x_3^{\alpha_3-2}$. If the tangent cone is 2-Buchsbaum, then $\alpha_2+\alpha_{12} \leq \alpha_1 + \alpha_3 - 1$. And then the algorithm stop at this step.

(c) If $x_1^2 x_3^{\alpha_3-1}$ is one minimal generate for I^*, then by the proof for 4.9, $\alpha_{12} = 1$, and the defining ideal is

$$I = (f_1 := x_1^{\alpha_1} - x_2x_3, f_2 := x_2^{\alpha_2} - x_1^2x_3^{\alpha_3-1}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_1-2}x_2^{\alpha_2-1}).$$

Similar to the previous case, the standard basis algorithm will only contribute an additional basis element $f_4 := \mathrm{spoly}(f_1, f_2) = x_2^{\alpha_2+1} - x_1^{\alpha_1+2}x_3^{\alpha_3-2}$.

□

In the rest of this section, we establish a new characterization on when the tangent cone $\mathrm{gr}_{\mathfrak{m}}(R)$ will be Cohen-Macaulay in the case $d = 3$. First of all, we give a remark to 2.7.

Remark 4.11. For symmetric G with arbitrary embedding dimension, if the order values of the Apéry set elements are symmetric in the sense of (†), and $\mathrm{s}_Q(\mathfrak{m}) = \mathrm{r}_Q(\mathfrak{m})$, then $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein, even without assuming it to be Cohen-Macaulay in the beginning. For the proof, see [6, 3.14].

We thank Lance Bryant for the helpful comments regarding 4.12.

Theorem 4.12. *If $G = \langle n_1, n_2, n_3\rangle$ is minimally 3-generated, and for the principal reduction $Q = (t^{n_1})R$ of the maximal ideal $\mathfrak{m}$, the index of nilpotency $\mathrm{s}_Q(\mathfrak{m})$ equals the reduction number $\mathrm{r}_Q(\mathfrak{m})$, then the tangent cone $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay.*

Proof. In the following, for $x \in S$, write $\bar{x}$ for its image in $R = S/I$. First we study the case when G is symmetric, i.e., the numerical semigroup ring R is a complete intersection. Now for $w_{e-1} = f + n_1$, $\mathrm{ord}_{\mathfrak{m}}(w_{e-1}) = \mathrm{s}_Q(\mathfrak{m})$ by 2.1.

Using 4.1 again, we have three cases.

(i) $(i, j, k) = (1, 2, 3)$, then the tangent cone is automatically a complete intersection.

(ii) $(i, j, k) = (1, 3, 2)$. Now the last Apéry set element can be written as

$$w_{e-1} = (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3.$$

This is obviously the unique representation of w_{e-1} with respect to G. Hence the order values of the Apéry set element is symmetric in the sense of (†). If $\mathrm{r}_Q(\mathfrak{m}) = \mathrm{s}_Q(\mathfrak{m})$, then $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein by 4.11.

(iii) $(i, j, k) = (2, 3, 1)$. Now the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if $\alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_3 - \alpha_{13}$. We want to prove that the following conditions are equivalent:

(a) $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay.
(b) $\mathrm{r}_Q(\mathfrak{m}) = \mathrm{s}_Q(\mathfrak{m})$.
(c) $\mathrm{r}_Q(\mathfrak{m}) \leq \alpha_1 + \alpha_3 - 2$.

Notice that the Frobenius number can be written as

$$f = (\alpha_2 + \alpha_{12} - 1)n_2 + (\alpha_{13} - 1)n_3 - n_1,$$

hence

$$w_{e-1} = (\alpha_2 + \alpha_{12} - 1)n_2 + (\alpha_{13} - 1)n_3.$$

This gives the maximal representation of w_{e-1} with respect to G, and $\mathrm{ord}_{\mathfrak{m}}(w_{e-1}) = (\alpha_2 + \alpha_{12} - 1) + (\alpha_{13} - 1)$.

That (a) implies (b) is clear.

If (b) holds, then for $r = \mathrm{r}_Q(\mathfrak{m})$, $\bar{x}_2^{\alpha_2+\alpha_{12}-1}\bar{x}_3^{\alpha_{13}-1} \in \mathfrak{m}^r$. So $\bar{x}_2^{\alpha_2+\alpha_{12}}\bar{x}_3^{\alpha_{13}-1} \in \mathfrak{m}^{r+1} = Q\mathfrak{m}^r$. $\bar{x}_2^{\alpha_2+\alpha_{12}}\bar{x}_3^{\alpha_{13}-1} = \bar{x}_2^{\alpha_{12}}\bar{x}_3^{\alpha_3+\alpha_{13}-1} = \bar{x}_1^{\alpha_1}\bar{x}_3^{\alpha_3-1}$ and $\bar{x_1}$ is a regular element in the domain R. Hence $\bar{x}_1^{\alpha_1-1}\bar{x}_3^{\alpha_3-1} \in \mathfrak{m}^r$. We want to show that $\mathrm{ord}_{\mathfrak{m}}(\bar{x}_1^{\alpha_1-1}\bar{x}_3^{\alpha_3-1}) = (\alpha_1 - 1) + (\alpha_3 - 1)$, hence (c) holds.

It suffices to show that $(\alpha_1-1)n_1+(\alpha_3-1)n_3$ is the unique representation of this element with respect to G. Suppose not, then

$$(\alpha_1 - 1)n_1 + (\alpha_3 - 1)n_3 = an_1 + bn_2 + cn_3,$$

with $a, b, c \in \mathbb{N}_0$ and $b > 0$. By the minimality of α_1 and α_3, one must have $a \leq \alpha_1 - 1$ and $c \leq \alpha_3 - 1$. Now

$$(\alpha_1 - 1 - a)n_1 + (\alpha_3 - 1 - c)n_3 = bn_2$$

Since $b > 0$, $b \geq \alpha_2$ by the minimality of α_2. Hence

$$(\alpha_1 - 1 - a)n_1 + (\alpha_3 - 1 - c)n_3 = (b - \alpha_2)n_2 + \alpha_3 n_3,$$

thus

$$(\alpha_1 - 1 - a)n_1 = (b - \alpha_2)n_2 + (c + 1)n_3,$$

which is against the minimality of α_1. This shows that (b) implies (c).

If (c) holds, then $\alpha_2 + \alpha_{12} + \alpha_{13} - 2 = \mathrm{s}_Q(\mathfrak{m}) \leq \mathrm{r}_Q(\mathfrak{m}) \leq \alpha_1 + \alpha_3 - 2$. It follows immediately that $\alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_3 - \alpha_{13}$. Hence $\mathrm{gr}_Q(\mathfrak{m})$ is Cohen-Macaulay and (a) holds.

Now we consider the case when the group G is not symmetric. Recall that the defining ideal shall be

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}).$$

Our aim is to show that if $\mathrm{s}_Q(\mathfrak{m}) = \mathrm{r}_Q(\mathfrak{m})$, then $\alpha_2 \leq \alpha_{21} + \alpha_{23}$. We first show that the index of nilpotency is

$$\mathrm{s}_Q(\mathfrak{m}) = \max\{\alpha_2 + \alpha_{13} - 2, \alpha_3 + \alpha_{12} - 2\}. \tag{1}$$

Notice that $\alpha_2 + \alpha_{13} - 2 = \alpha_2 + \alpha_3 - 2 - \alpha_{23}$, and $\alpha_3 + \alpha_{12} - 2 = \alpha_2 + \alpha_3 - 2 - \alpha_{32}$. $\mathrm{s}_Q(\mathfrak{m}) = \max\{\mathrm{ord}_{\mathfrak{m}}(w) \mid w \in \mathrm{Ap}(G, e)\}$. Every $w \in \mathrm{Ap}(G, e)$ can be written as $w = bn_2 + cn_3$ for some $b, c \in \mathbb{N}_0$. Obviously, $b < \alpha_2$ and $c < \alpha_3$. And it cannot happen that both $b \geq \alpha_{12}$ and $c \geq \alpha_{13}$. Hence for w with this representation, if $b = \alpha_2 - 1 \geq \alpha_{12}$, then $c < \alpha_{13}$ because of f_1. Similarly, if $c = \alpha_3 - 1 \geq \alpha_{13}$, then $b < \alpha_{12}$ because of f_2. On the other hand, it is not difficult to see that $(\alpha_2-1)n_2+(\alpha_{13}-1)n_3$ and $(\alpha_{12}-1)n_2+(\alpha_3-1)n_3$ are elements of $\mathrm{Ap}(G, e)$. For instance, suppose for contradiction that $(\alpha_2-1)n_2+(\alpha_{13}-1)n_3 = an_1+bn_2+cn_3$

with $a, b, c \in \mathbb{N}_0$ and $a > 0$. By the minimality of α_2 and α_3, it is clear that $\alpha_2 - 1 \geq b$ and $\alpha_{13} - 1 \geq c$. Now

$$(\alpha_2 - 1 - b)n_2 + (\alpha_{13} - 1 - c)n_3 = an_1.$$

$a > 0$, hence $a \geq \alpha_1$. Now

$$(\alpha_2 - 1 - b)n_2 + (\alpha_{13} - 1 - c)n_3 = (a - \alpha_1)n_1 + \alpha_{12}n_2 + \alpha_{13}n_3.$$

This implies

$$(\alpha_{32} - 1 - b)n_2 = (a - \alpha_1)n_1 + (c + 1)n_3.$$

This contradicts the minimality of α_2. One can argue in a similar way for $(\alpha_{12} - 1)n_2 + (\alpha_3 - 1)n_3$. Now the formula (1) follows naturally.

The case when $\mathrm{s}_Q(\mathfrak{m}) = \alpha_2 + \alpha_{13} - 2$ is easy. Suppose the condition is satisfied, i.e., $\alpha_2 + \alpha_{13} - 2 = r = \mathrm{r}_Q(\mathfrak{m})$. Then $x_2^{\alpha_2} x_3^{\alpha_{13}-1} \in \mathfrak{m}^{r+1} = Q\mathfrak{m}^r$. Notice that $x_2^{\alpha_2} x_3^{\alpha_{13}-1} = x_1^{\alpha_{21}} x_3^{\alpha_3 - 1}$. Hence $x_1^{\alpha_{21}-1} x_3^{\alpha_3-1} \in \mathfrak{m}^r$. Similar to the proof for (3) of Gorenstein case, one can show that this decomposition is unique, hence $\mathrm{ord}_{\mathfrak{m}}(x_1^{\alpha_{21}-1} x_3^{\alpha_3-1}) = \alpha_{21} + \alpha_3 - 2 \geq r = s = \alpha_2 + \alpha_{13} - 2$. Hence $\alpha_2 \leq \alpha_{21} + \alpha_3 - \alpha_{13} = \alpha_{21} + \alpha_{23}$.

If $\mathrm{s}_Q(\mathfrak{m}) > \alpha_2 + \alpha_{13} - 2$ and $\mathrm{r}_Q(\mathfrak{m}) = \mathrm{s}_Q(\mathfrak{m})$, then $\delta := \alpha_{23} - \alpha_{32} > 0$ and $\alpha_2 + \alpha_{13} - 2 = r - \delta$. Now $\mathrm{ord}_{\mathfrak{m}}(x_2^{\alpha_2 - 1} x_3^{\alpha_{13}-1}) \geq r - \delta$, hence $x_2^{\alpha_2+\delta} x_3^{\alpha_{13}-1} \in \mathfrak{m}^{r+1} = Q\mathfrak{m}^r$. It follows easily that $g_0 := x_1^{\alpha_{21}-1} x_2^{\delta} x_3^{\alpha_3-1} \in \mathfrak{m}^r$. It is not true that this is the unique representation. However, suppose that associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay, then $\alpha_2 = \alpha_{12} + \alpha_{32} > \alpha_{21} + \alpha_{23}$. Hence $\delta = \alpha_{23} - \alpha_{32} < \alpha_{12} - \alpha_{21} < \alpha_{12}$. If we can modify the representation of g_0, then the smallest possible step to do it for the time being will be to use f_3. Now g takes a different representation $g_1 := x_1^{\alpha_{21}-\alpha_{31}-1} x_2^{\delta - \alpha_{32}} x_3^{2\alpha_3 - 1}$. It is by the same reason that the only minimal relation that can be possibly used to modify the representation of g_1 is f_3. And either you go back to get g_0, or you continue to drop the exponents of x_1 and x_2, but increase the exponent of x_3. This argument can continue, i.e., one can only use f_3 for a multiple of times to modify the representation of g_0 to get some new g_i. Notice that $\alpha_3 < \alpha_{31} + \alpha_{32}$, hence $\mathrm{ord}_{\mathfrak{m}}(g_i)$ is achieved when it is written as g_0. The rest of the proof is similar. $\mathrm{ord}_{\mathfrak{m}}(g_0) = \delta + \alpha_{21} + \alpha_3 - 2 \geq r = s = \alpha_2 + \alpha_{13} - 2 + \delta$. Hence $\alpha_2 \leq \alpha_{21} + \alpha_3 - \alpha_{13} = \alpha_{21} + \alpha_{23}$. □

The previous theorem fails if the embedding dimension is 4.

Example 4.13. Let $G = \langle 9, 10, 11, 23 \rangle$. Then G is symmetric, and $\mathrm{s}_Q(\mathfrak{m}) = \mathrm{r}_Q(\mathfrak{m}) = 4$. But $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay.

If the 1-dimensional local ring R is not associated to numerical semigroup, then theorem might still fail, even when it has embedding dimension 3. The prototype of the following example is due to Lance Bryant.

Example 4.14. Let $S = \mathbb{C}[[a, b, c]]$ and $R = S/I$ where $I = (a^3 + c^5 + b^6, a^2b + ac^3 + b^6)$. Then R is a 1-dimensional reduced ring. The initial form ideal $I^* = (b^2c^5 + ac^6, abc^5, a^2c^3, a^2b, a^3)$, hence $\mathbb{C}[a, b, c]/I^*$ is not Cohen-Macaulay. On the other hand, $Q = (b-c)R$ is a principal reduction of the maximal ideal $\mathfrak{m} = (a, b, c)R$. And $\mathrm{r}_Q(\mathfrak{m}) = \mathrm{s}_Q(\mathfrak{m}) = 6$.

5. When the embedding dimension $d = 4$

In this section, we want to prove the following main result by studying the standard basis of the corresponding defining ideal.

Theorem 5.1. *Let the numerical semigroup $G = \langle n_1, n_2, n_3, n_4\rangle$ be minimally generated by 4 elements. If the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein, then for every $x \in \mathrm{Ap}(G, n_1)$, $x \preceq_G (f + n_1)$.*

The proof of the above theorem depends heavily on an important result by Bresinsky [3]:

Theorem 5.2. *Let $G = \langle n_1, n_2, n_3, n_4\rangle$ be a symmetric numerical semigroup, minimally generated by 4 elements. Then up to a permutation of generators, the defining ideal I can be classified into the following three cases.*

- (I) $I = (x_1^{\alpha_1} - x_2^{\alpha_2}, x_3^{\alpha_3} - x_4^{\alpha_4}, x_1^{\beta_1}x_2^{\beta_2} - x_3^{\beta_3}x_4^{\beta_4})$.
- (II) $I = (x_1^{\alpha_1} - x_2^{\alpha_2}, x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}, x_4^{\alpha_4} - x_1^{\alpha_{41}}x_2^{\alpha_{42}}x_3^{\alpha_{43}})$ *with* $\alpha_{ij} \in \mathbb{N}_0$.
- (III) $I = \{x_1^{\alpha_1} - x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}}, x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}, x_4^{\alpha_4} - x_2^{\alpha_{42}}x_3^{\alpha_{43}}, x_3^{\alpha_{43}}x_1^{\alpha_{21}} - x_2^{\alpha_{32}}x_4^{\alpha_{14}}\}$ *with* $0 < \alpha_{ij} < \alpha_j$.

Remark 5.3. Let $T = k[x_1, \cdots, x_d]$ be a polynomial ring of dimension d over the field k. Let $\mathfrak{m} = (x_1, \cdots, x_d)$ be the graded maximal ideal of T. If an T-ideal J is $\mathfrak{m}$-primary, monomial and Gorenstein, then it is generated by pure powers of the form $\{x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}\}$. The proof is an easy application of linkage theory. When the ideal is Gorenstein, one can use the generators that are pure powers to generate a link. The resulting quotient ideal is an almost complete intersection and monomial. Now use the same pure powers to generate a new link, and we shall get the original monomial ideal. But it will be easy to see that for an almost complete intersection monomial ideal, this link has to be generated by pure powers. See [13] for related discussion.

As a generalization of the above idea, we can prove the following:

Proposition 5.4. *Let ideal $L \subset S = k[x, y, z]$ be generated by $f = x^a - y^{b'}z^{c'} \in (x, y, z)$ and a finite collection of monomials. If $\mathrm{ht}(L) = 3$ and L is Gorenstein, then L is generated by up to 5 elements.*

Proof. In the following, fix the binomial f. We say an ideal K is *almost monomial* if we can write $K = (f) + K'$ where K' is an monomial ideal in $k[x, y, z]$. We fix a monomial ordering so that the initial monomial of f is x^a. Depending on whether $a \geq b' + c'$ or not, this monomial ordering is either global or local. Applying the reduced standard basis algorithm to the existing generators of I, we may have a minimal standard basis $\{f_1, \ldots, f_s\}$. At most one of the f_i is a binomial; the remaining are monomials. If f is part of this minimal standard basis, then we say K is *strictly almost monomial*. Following 5.3, we may always assume that L is strictly almost monomial.

Since L has the maximal possible height, we can write L in a standard form $L = (f, y^b, z^c) + L^\#$, where $L^\#$ is (minimally) generated by the remaining monomial generators of the standard basis. Let $J = (f, y^b, z^c)$, then

$$J : L = J : L^\# = \bigcap_{x^\alpha y^\beta z^\gamma \in L^\#} J : x^\alpha y^\beta z^\gamma.$$

Notice that for every $x^\alpha y^\beta z^\gamma \in S$

$$\begin{aligned} J : x^\alpha y^\beta z^\gamma =&(f, y^{\max(b-\beta,0)}, z^{\max(c-\gamma,0)}, \\ &x^{\max(a-\alpha,0)} y^{\max(b-b'-\beta,0)}, x^{\max(a-\alpha,0)} z^{\max(c-c'-\gamma,0)}). \end{aligned} \tag{2}$$

To prove it, one can argue directly or use standard basis arguments which we give in the Appendix.

Hence the quotient ideal $J : L$ is an intersection of almost monomial ideals. We want to show that it is again almost monomial. This can be done by using induction. Suppose $K_i = (f) + K'_i$, $i = 1, 2$, are two almost monomial ideals, then

$$K_1 \cap K_2 = (f) + (K'_1 \cap K'_2). \tag{3}$$

To see this, observe that for every element $g \in S$, there is $g' = \mathrm{NF}(g|\,\{f\}) \in S$. For every monomial $m \in$ mono-Supp(g'), m is not divisible by x^a. Hence $g \in K$ if and only if $g' \in K'$.

Since L is Gorenstein, $\mu((J : L)/J) = 1$. Hence the almost monomial ideal $J : L = J + x^\alpha y^\beta z^\gamma$ for some monomial $x^\alpha y^\beta z^\gamma \in S$. And $L = J : x^\alpha y^\beta z^\gamma$ is given by the formula (2). In particular, L is generated by up to 5 elements. □

Corollary 5.5 (Structure for Gorenstein strictly almost monomial ideals)**.** *Let $L \subset k[x, y, z]$ be a strictly almost monomial ideal with $f = x^a - y^{b'} z^{c'}$. Then L is Gorenstein and not a complete intersection if and only if L can written as*

$$L = (f, y^b, z^c, x^{a-\alpha} y^{b-b'}, x^{a-\alpha} z^{c-c'}). \tag{4}$$

Proof. If L is almost monomial and Gorenstein, but not complete intersection, then we can write L in the format (2). Since we can assume that y^b and z^c belong to the minimal generators of L, one must have $\beta = \gamma = 0$ and L can be written in the given format (4).

On the other hand, if L is written as in (4), then it is generated by the five submaximal Pfaffians of the anti-symmetric matrix

$$M = \begin{pmatrix} 0 & 0 & -y^{b'} & 0 & x^{a-\alpha} \\ 0 & 0 & -z^{c-c'} & y^{b-b'} & 0 \\ y^{b'} & z^{c-c'} & 0 & x^\alpha & 0 \\ 0 & -y^{b-b'} & -x^\alpha & 0 & z^{c'} \\ -x^{a-\alpha} & 0 & 0 & -z^{c'} & 0 \end{pmatrix}.$$

Since $\mathrm{ht}(L) = 3$ and L is Cohen-Macaulay, by the Buchsbaum-Eisenbud's theorem [5, 3.4.1], L is Gorenstein. □

Remark 5.6. With the assumptions as in Corollary 5.5 and equip the polynomial ring $k[x, y, z]$ with a monomial ordering, such that the leading monomial of f is x^a. Let L be Gorenstein and strictly almost monomial with standard basis $\{f, y^b, z^c, g_4, \ldots, g_s\}$, where all the g_i's are monomials. One observes the following from (4).

(a) None of the g_i's is divisible by yz.
(b) If L is not a complete intersection, then it has the two additional monomial generators that are not pure powers. These two monomials share the same x component.

(c) None of the g_i's is a pure power in x. In other words, If a pure power term in x is a minimal generator for a Gorenstein almost monomial ideal, then this ideal is generated by pure powers in x, y and z.

These observations will play an essential role in the proof of 5.1.

Remark 5.7. Return to our discussion of numerical semigroup ring R. When the embedding dimension $d = 4$, we know $\mathrm{ht}(I) = \mathrm{ht}(I^*) = 3$. Since $\sqrt{I^*} = (x_2, x_3, x_4)$, $\mathrm{ht}(I^* + (x_1)) = 4$. Thus if we assume that I^* is Cohen-Macaulay, then x_1 is regular modulo I^*. And I^* is Gorenstein if and only if $I^* + (x_1)$ is so. Furthermore, if $f_1, \cdots, f_m$ is a minimal standard basis for I with respect to suitable local ordering, then the initial forms $f_1^*, \cdots, f_m^*$ minimally generate I^*. For each i, if f_i^* is a monomial, since I^* is Cohen-Macaulay, f_i^* does not involve any x_1 term by 3.2. Let $\tilde{}$ denote the image by modulo x_1. If $\widetilde{f_i^*}$ is a binomial, then $f_i = x_3^{\beta_3} - x_2^{\beta_2} x_4^{\beta_4}$ is balanced. On the other hand, it is easy to check that at most one $\widetilde{f_j}$ can be binomial. Hence $\widetilde{I^*}$ is almost monomial where x_3 plays the same role as x in 5.5.

It follows from the previous discussion that

Corollary 5.8. *If* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein, then* $\mu(I^*) \leq 5$.

Now we are ready for the proof of 5.1. We proceed according to the three cases in 5.2. We always assume the monomial order for $T = k[x_1, x_2, x_3, x_4] \cong \mathrm{gr}_{\mathfrak{n}}(S)$ is first nice in x_1, then nice in x_2. For instance, the usual negative degree reverse lexicographical ordering on $k[x_4, x_3, x_2, x_1]$ satisfies this requirement. In particular, if $x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}$ is balanced with $\alpha_{32} > 0$, its initial monomial is $x_3^{\alpha_3}$.

5.1. **Proof for Case I.**

Remark 5.9. The original statement of Theorem 4 in [3] contains an unnecessary condition $0 < \beta_i < \alpha_i$. This restriction is incorrect, as can be seen in 5.11. But the proof of it still holds without any further change. And the Lemma 5 before it only needs minimal modifications. It follows from this Lemma 5 and its corollary that

$$f + n_1 = (\alpha_2 - 1)n_2 + (\alpha_3 + \beta_3 - 1)n_3 + (\beta_4 - 1)n_4. \tag{5}$$

Proof for case I. If we insist that $n_1 < n_2 < n_3 < n_4$, then by a permutation we need to consider the following three sub-cases:

(i) $I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_2}, f_2 := x_3^{\alpha_3} - x_4^{\alpha_4}, f_3 := x_1^{\beta_1} x_2^{\beta_2} - x_3^{\beta_3} x_4^{\beta_4})$;
(ii) $I = (f_1 := x_1^{\alpha_1} - x_3^{\alpha_3}, f_2 := x_2^{\alpha_2} - x_4^{\alpha_4}, f_3 := x_1^{\beta_1} x_3^{\beta_3} - x_2^{\beta_2} x_4^{\beta_4})$;
(iii) $I = (f_1 := x_1^{\alpha_1} - x_4^{\alpha_4}, f_2 := x_2^{\alpha_2} - x_3^{\alpha_3}, f_3 := x_1^{\beta_1} x_4^{\beta_4} - x_2^{\beta_2} x_3^{\beta_3})$.

Let's prove according to these three cases.

(i) Due to the generator f_2, we may assume that $0 \leq \beta_4 \leq \alpha_4 - 1$ in f_3. Hence $\beta_3 \neq 0$ by the choice of α_4. Similarly, we assume that $0 \leq \beta_2 \leq \alpha_2 - 1$, and hence $\beta_1 \neq 0$. Since $\mathrm{spoly}(f_1, f_2)$ does not contribute to the standard basis, the initial forms $f_1^* = -x_2^{\alpha_2}$, $f_2^* = -x_4^{\alpha_4}$, $f_3^* = -x_3^{\beta_3} x_4^{\beta_4}$ form part of the minimal basis of I^*.

By 5.7 and 5.6, $\tilde{I}^*$ is almost monomial. If $\beta_4 \neq 0$, then $\tilde{I}^*$ is strictly almost monomial. Now by applying standard basis algorithm, we get $f_4 = \mathrm{spoly}(f_2, f_3) = x_3^{\alpha_3+\beta_3} - x_1^{\beta_1} x_2^{\beta_2} x_4^{\alpha_4 - \beta_4}$. By the Cohen-Macaulay condition, we must have $\alpha_3 + \beta_3 \leq \beta_1 + \beta_2 + \alpha_4 - \beta_4$. Since $\beta_1 \neq 0$, $\tilde{I}^*$ has one minimal generator $x_3^{\alpha_3+\beta_3}$ which is a pure power. This is impossible for

strictly almost monomial ideals. Hence $\beta_4 = 0$ and I^* is generated by pure powers. Notice that f_1, f_2 and f_3 form a standard basis for I, and $\beta_3 \geq \alpha_3$. It follows from the formula (5) that

$$f + n_1 = (\alpha_2 - 1)n_2 + (\beta_3 - 1)n_3 + (\alpha_4 - 1)n_4. \tag{6}$$

The decomposition of $f + n_1$ with respect to G might not be unique in general. But it is easy to see that the coefficient $(\alpha_2 - 1)$ of n_2 is fixed. Now if $\beta_3 = \alpha_3$, then the above representation is unique. Otherwise, $\beta_3 - 1 \geq \alpha_3$, and we have at least one extra representation

$$f + n_1 = (\alpha_2 - 1)n_2 + (\beta_3 - 1 - \alpha_3)n_3 + (2\alpha_4 - 1)n_4.$$

Since $n_3 < n_4$, we have $\alpha_3 > \alpha_4$ by the relation $\alpha_3 n_3 = \alpha_4 n_4$. Hence the decomposition in (6) is the unique one such that gives us the maximal length of $f + n_1$ in terms of n_2, n_3 and n_4. Now $\operatorname{ord}_G(f + n_1) = (\alpha_2 - 1) + (\beta_3 - 1) + (\alpha_4 - 1)$.

Similarly, every element $z_2n_2 + z_3n_3 + z_4n_4 \in \operatorname{Ap}(G, n_1)$ can be written in such a way that $0 \leq z_2 \leq \alpha_2 - 1$, $0 \leq z_3 \leq \beta_3 - 1$ and $0 \leq z_4 \leq \alpha_4 - 1$. $\operatorname{ord}_G(z_2n_2 + z_3n_3 + z_4n_4) = z_1 + z_2 + z_3$.

In particular, we have

$$(z_2n_2 + z_3n_3 + z_4n_4) \preceq_G (f + n_1).$$

(ii) We may assume that $0 \leq \beta_3 \leq \alpha_3 - 1$ and $0 \leq \beta_4 \leq \alpha_4 - 1$, so that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. We have $f_1^* = -x_3^{\alpha_3}$ and $f_2^* = -x_4^{\alpha_4}$. By comparing $\beta_1 + \beta_3$ with $\beta_2 + \beta_4$, we have three cases for f_3^*.

(a) $f_3^* = x_1^{\beta_1} x_3^{\beta_3}$. Following 5.7, the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ won't be Cohen-Macaulay unless $\beta_1 = 0$, which is impossible here.

(b) $f_3^* = -x_2^{\beta_2} x_4^{\beta_4}$. Since $\tilde{I}^*$ is almost monomial, if I^* is Gorenstein, then $\beta_4 = 0$. Now $\beta_2 < \beta_1 + \beta_3$. Thus considering the permutation, from (5) we have

$$f + n_1 = (\beta_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4.$$

Similar to the discussion in case (i), this decomposition is the unique maximal one. The rest of the proof is also similar.

(c) $f_3^* = x_1^{\beta_1} x_3^{\beta_3} - x_2^{\beta_2} x_4^{\beta_4}$. After modulo x_1, we are reduced to a case similar to (b).

(iii) $f_2^* = -x_3^{\alpha_3}$ is a pure power in x_3. Hence $\tilde{I}^*$ is a complete intersection in pure powers. This implies that $\beta_3 = 0$ and $\beta_2 \leq \beta_4 + \beta_1$ if we assume that $0 \leq \beta_4 \leq \alpha_4 - 1$ from the beginning. The remaining discussion is similar to that of case (i).

□

Corollary 5.10. *In case I of 5.2, if $\operatorname{gr}_{\mathfrak{m}}(R)$ is Gorenstein, then it is a complete intersection.*

To illustrate the above proposition, we give several examples.

Example 5.11. (i) If $G = \langle 8, 12, 14, 21 \rangle$, then the defining ideal $I = (x_4^2 - x_3^3, x_3^2 - x_1^2 x_2, x_2^2 - x_1^3)$ and the initial form ideal $I^* = (x_2^2, x_3^2, x_4^2)$.

(ii) If $G = \langle 8, 10, 12, 15 \rangle$, then $I = (x_4^2 - x_2^3, x_3^2 - x_1^3, x_2^2 - x_1x_3)$ and $I^* = (x_4^2, x_3^2, x_2^2 - x_1x_3)$.

(iii) If $G = \langle 30, 33, 44, 45\rangle$, then $I = (x_2^5 - x_1^4x_4, x_3^3 - x_2^4, x_4^2 - x_1^3)$ and $I^* = (x_2^5 - x_1^4x_4, x_3^3, x_4^2)$.

5.2. **Proof for Case II.** In this case, the defining ideal is the complete intersection ideals studied in [11]. In particular, by [11, 2.1] the Frobenius number

$$f = (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4 - n_1. \tag{7}$$

We prove in accordance with the permutation (i, j, k, l) of $(1, 2, 3, 4)$. By symmetry, we can always conveniently assume that $i < j$, and consider the following 12 cases.

(i) $(i, j, h, k) = (1, 2, 3, 4)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_2}, f_2 := x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}, f_3 := x_4^{\alpha_4} - x_1^{\alpha_{41}}x_2^{\alpha_{42}}x_3^{\alpha_{43}}).$$

Since $n_1 < n_2 < n_3 < n_4$, we have $\alpha_2 < \alpha_1$, $\alpha_3 < \alpha_{31} + \alpha_{32}$ and $\alpha_4 < \alpha_{41} + \alpha_{42} + \alpha_{43}$. Now for every $x \in \mathrm{Ap}(G, n_1)$, suppose $x = a_2n_2 + a_3n_3 + a_4n_4$ with $a_2 + a_3 + a_4 = \mathrm{ord}_G(x)$. By the maximality of $\mathrm{ord}_G(x)$, one must have $a_i < \alpha_i$ for $i = 2, 3, 4$. On the other hand, by (7),

$$f + n_1 = (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4.$$

Now it follows immediately that $x \preceq_G (f + n_1)$.

(ii) $(i, j, h, k) = (1, 2, 4, 3)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_2}, f_2 := x_4^{\alpha_4} - x_1^{\alpha_{31}}x_2^{\alpha_{42}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}x_4^{\alpha_{34}}).$$

One has $\alpha_2 < \alpha_1$ and $\alpha_4 < \alpha_{41} + \alpha_{42}$ automatically. Now we can assume that in f_3, $0 \le \alpha_{32} < \alpha_2$ and $0 \le \alpha_{34} < \alpha_3$. If $\alpha_{31} \ne 0$, then by the Cohen-Macaulay condition, $\alpha_3 \le \alpha_{31} + \alpha_{32} + \alpha_{34}$. If $\alpha_{31} = 0$, by our previous assumption on α_{32} and α_{34}, f_3^* is part of the minimal generators for I^*. But if I^* is Gorenstein, terms of the form $x_2^{\beta_2}x_4^{\beta_4}$ with $\beta_2, \beta_4 > 0$ will not be part of the minimal generators. Hence either $\alpha_3 \le \alpha_{32} + \alpha_{34}$ or one of α_{32} and α_{34} is zero. The latter cannot happen, since it is against to the choice of α_4 or α_2 respectively. In short, if we assume that I^* is Gorenstein, then we should have $\alpha_3 \le \alpha_{31} + \alpha_{32} + \alpha_{34}$. And the rest of the proof is similar to that of (i).

(iii) $(i, j, h, k) = (1, 3, 2, 4)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_3^{\alpha_3}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, f_3 := x_4^{\alpha_4} - x_1^{\alpha_{41}}x_2^{\alpha_{42}}x_3^{\alpha_{43}}).$$

We automatically have $\alpha_3 < \alpha_1$ and $\alpha_4 < \alpha_{41} + \alpha_{42} + \alpha_{43}$. We can always assume that $0 \le \alpha_{23} < \alpha_3$. Hence the initial form f_2^* is part of the minimal generators for I^*. The Cohen-Macaulay condition would force $\alpha_2 \le \alpha_{21} + \alpha_{23}$. The rest of the proof is similar to that of (i).

(iv) $(i, j, h, k) = (1, 3, 4, 2)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_3^{\alpha_3}, f_2 := x_4^{\alpha_4} - x_1^{\alpha_{41}}x_3^{\alpha_{43}}, f_3 := x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}x_4^{\alpha_{24}}).$$

We have $\alpha_3 < \alpha_1$ and $\alpha_4 < \alpha_{41} + \alpha_{43}$. In addition, we can assume that $0 \le \alpha_{23} < \alpha_3$ and $0 \le \alpha_{24} < \alpha_4$. Now f_3^* is part of the minimal generators. Notice that $f_1^* = -x_3^{\alpha_3}$ is also one such minimal generator. Hence the Gorenstein ideal I^* after modulo x_1 has to be a complete intersection ideal generated by pure powers. This would force that $\alpha_2 \le \alpha_{21} + \alpha_{23} + \alpha_{24}$. The rest of the proof is similar to that of (i).

(v) $(i,j,h,k) = (1,4,2,3)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_4^{\alpha_4}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}} x_4^{\alpha_{34}}).$$

We have $\alpha_4 < \alpha_1$. Now we can assume that $0 \le \alpha_{24}, \alpha_{34} < \alpha_4$. If f_2^* is part of a system of minimal generators, then the Cohen-Macaulayness would require that $\alpha_2 \le \alpha_{21} + \alpha_{24}$. The rest of the proof is similar to that of (ii).

If f_2^* is not part of the minimal generating set for I^*, then $\alpha_{32} = 0$, $\alpha_3 > \alpha_{31} + \alpha_{34}$, $\alpha_2 > \alpha_{21} + \alpha_{24}$ and $x_1^{\alpha_{21}} x_4^{\alpha_{24}}$ is divisible by $x_1^{\alpha_{31}} x_4^{\alpha_{34}}$. The problem will be reduced to the case (vi) with $\alpha_{23} > 0$.

(vi) $(i,j,h,k) = (1,4,3,2)$. Then

$$I = (f_1 := x_1^{\alpha_1} - x_4^{\alpha_4}, f_2 := x_3^{\alpha_3} - x_1^{\alpha_{31}} x_4^{\alpha_{34}}, f_3 := x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} x_4^{\alpha_{24}}).$$

Clearly $\alpha_4 < \alpha_1$ and we can assume that $0 \le \alpha_{24}, \alpha_{34} < \alpha_4$. Now if initial form f_2^* belongs to a system of minimal generators for I^*, then the Cohen-Macaulay condition implies $\alpha_3 \le \alpha_{31} + \alpha_{34}$. The rest of the proof is similar to that of (iv).

If f_2^* does not belong to a system of minimal generators for I^*, then $\alpha_{23} = 0$, $\alpha_2 > \alpha_{21} + \alpha_{24}$, $\alpha_3 > \alpha_{31} + \alpha_{34}$ and $x_1^{\alpha_{31}} x_4^{\alpha_{34}}$ is divisible by $x_1^{\alpha_{21}} x_4^{\alpha_{24}}$. Now the problem is reduced to the case (v) with $\alpha_{32} > 0$.

(vii) $(i,j,h,k) = (2,3,1,4)$. Then

$$I = (f_1 := x_2^{\alpha_2} - x_3^{\alpha_3}, f_2 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, f_3 := x_4^{\alpha_4} - x_1^{\alpha_{41}} x_2^{\alpha_{42}} x_3^{\alpha_{43}}).$$

Since $n_1 < n_2 < n_3 < n_4$, $f_1^* = -x_3^{\alpha_3}$, $f_2^* = -x_2^{\alpha_{12}} x_3^{\alpha_{13}}$ and $f_3^* = x_4^{\alpha_4}$. We can always assume that $0 \le \alpha_{13} < \alpha_3$. Those f_i^* belong to a system of minimal generators for I^*. Notice that f_1^* is a pure power in x_3. Hence after modulo x_1, I^* is generated by pure powers in x_2, x_3 and x_4. f_2^* does not involve x_1 term, hence need to be a pure power. One cannot have $\alpha_{12} = 0$, since this is against the choice of α_{13} and α_3. Hence $\alpha_{13} = 0$ and $\alpha_{12} \ge \alpha_2$.

Now for every $x \in \operatorname{Ap}(G, n_1)$, we can write $x = \sum_i a_i n_i$ with $\sum_i a_i = \operatorname{ord}_G(x)$. Obviously $a_1 = 0$, $a_2 < \alpha_{12}$, $a_3 < \alpha_3$ and $a_4 < \alpha_4$. Meanwhile, by (7),

$$f + n_1 = (\alpha_{12} - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4.$$

Clearly $x \preceq_G (f + n_1)$.

(viii) $(i,j,h,k) = (2,3,4,1)$. Then

$$I = (f_1 := x_2^{\alpha_2} - x_3^{\alpha_3}, f_2 := x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_3 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}}).$$

Obviously $f_1^* = -x_3^{\alpha_3}$, $f_2^* = x_4^{\alpha_4}$ and $f_3^* = -x_2^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}}$. In the f_3, using f_1 and f_2, we can assume that $0 \le \alpha_{13} < \alpha_3$ and $0 \le \alpha_{14} < \alpha_4$. The rest of the proof is similar to that of (vii). In particular, we have $\alpha_{13} = \alpha_{14} = 0$.

(ix) $(i,j,h,k) = (2,4,1,3)$. Then

$$I = (f_1 := x_2^{\alpha_2} - x_4^{\alpha_4}, f_2 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_4^{\alpha_{14}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}} x_4^{\alpha_{34}}).$$

$f_1^* = -x_4^{\alpha_4}$ and $f_2^* = -x_2^{\alpha_{12}} x_4^{\alpha_{14}}$. Using f_1 we can always assume that $0 \le \alpha_{14}, \alpha_{34} < \alpha_4$. Using f_2, we can assume that $x_2^{\alpha_{12}} x_4^{\alpha_{14}}$ does not divide $x_1^{\alpha_{31}} x_2^{\alpha_{32}} x_4^{\alpha_{34}}$. Notice that if $\operatorname{gr}_{\mathfrak{m}}(R)$ is Gorenstein, then after modulo x_1, I^* is an almost monomial ideal where $x_2^{\beta_2} x_4^{\beta_4}$ is not part of the minimal

generators, for $\beta_2, \beta_4 > 0$. Hence either f_2^* is not part of the minimal generators or it is a pure power.

The first case happens if and only if $\alpha_{31} = 0$, $\alpha_{12} \geq \alpha_{32}$, $\alpha_{14} \geq \alpha_{34}$ and $\alpha_3 > \alpha_{32} + \alpha_{34}$. Then this problem can be reduced to the case (x) with $\alpha_{13} > 0$.

For the second situation, since $\alpha_{14} < \alpha_4$, f_2^* is a pure power if and only if $\alpha_{14} = 0$ and $\alpha_{12} \geq \alpha_2$. Now in addition, one may assume that $0 \leq \alpha_{32} \leq \alpha_{12} - 1$. For f_3, if $\alpha_3 \leq \alpha_{31} + \alpha_{32} + \alpha_{34}$, the proof can continue as in (i). This condition must be satisfied when $\alpha_{31} \neq 0$ and $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. Otherwise, assume $\alpha_{31} = 0$ and $\alpha_3 > \alpha_{32} + \alpha_{34}$. Then $\alpha_{32}, \alpha_{34} > 0$ and $f_3^* = -x_2^{\alpha_{32}} x_4^{\alpha_{34}}$ will be part of the minimal basis of I^*. Whence $\mathrm{gr}_{\mathfrak{m}}(R)$ will not be Gorenstein.

(x) $(i,j,h,k) = (2,4,3,1)$. Then

$$I = (f_1 := x_2^{\alpha_2} - x_4^{\alpha_4}, f_2 := x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, f_3 := x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}}).$$

We can assume that $0 \leq \alpha_{14}, \alpha_{34} < \alpha_4$.

(A) If $\alpha_3 < \alpha_{32} + \alpha_{34}$, we can also assume that $0 \leq \alpha_{13} < \alpha_3$. Now f_1^*, f_2^* and $f_3^* = -x_2^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}}$ form part of a system of minimal generators for I^*. f_2^* is a pure power in x_3, hence after modulo x_1, I^* is a complete intersection generated by pure powers as well. Thus $\alpha_{13} = \alpha_{14} = 0$ and the rest of the proof is similar to that of (i).

(B) If $\alpha_3 > \alpha_{32} + \alpha_{34}$, like in (ix), either we have $\alpha_{34} = 0$ and we have a contradiction, or $\alpha_{13} = 0$ and we can reduce the problem to the (ix) with $\alpha_{31} > 0$.

(C) If $\alpha_3 = \alpha_{32} + \alpha_{34}$, then under the monomial ordering we specified early this section, the initial monomial of f_2 is $x_3^{\alpha_3}$. Now we further assume that $0 \leq \alpha_{13} < \alpha_3$. Hence f_1^*,f_2^* and f_3^* form part of the system of generators for I^*, which is an almost monomial ideal when modulo x_1. By the structure of such ideals, $-f_3^*$ is $x_2^{\alpha_{12}}$, or $x_2^{\alpha_{12}} x_3^{\alpha_{13}}$, or $x_3^{\alpha_{13}} x_4^{\alpha_{14}}$, with strictly positive exponents.

(a) If $-f_3^* = x_2^{\alpha_{12}}$, we can follow the proof in (i).

(b) If $-f_3^* = x_2^{\alpha_{12}} x_3^{\alpha_{13}}$, we apply standard basis algorithm to find the standard basis for I. During the process, we have $\mathrm{spoly}(f_2, f_3) = x_2^{\alpha_{32}+\alpha_{12}} x_4^{\alpha_{34}} - x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_{13}}$. It is part of a minimal standard basis for I. Hence according to the structure for Gorenstein almost monomial ideal, $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Gorenstein.

(c) If $-f_3^* = x_3^{\alpha_{13}} x_4^{\alpha_{14}}$, by the structure property again, one has $\alpha_{14} + \alpha_{34} = \alpha_4$. Applying the standard basis algorithm, we find the two extra generators $f_4 := x_2^{\alpha_2} x_3^{\alpha_{13}} - x_1^{\alpha_1} x_4^{\alpha_{34}}$ and $f_5 := x_2^{\alpha_2+\alpha_{32}} - x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_{13}}$. In particular, we have the necessary conditions $\alpha_2 + \alpha_{13} \leq \alpha_1 + \alpha_{34}$ and $\alpha_2 + \alpha_{32} \leq \alpha_1 + \alpha_3 - \alpha_{13}$. Now for every $x = \sum a_i n_i \in \mathrm{Ap}(G, n_1)$ with $\sum a_i = \mathrm{ord}_G(x)$, $a_1 = 0$, $a_2 < \alpha_2 + \alpha_{32}$ and $a_4 < \alpha_4$. Furthermore, $x_2^{\alpha_2} x_3^{\alpha_{13}}$ does not divide $x_2^{a_2} x_3^{a_3}$ and $x_3^{\alpha_{13}} x_4^{\alpha_{14}}$ does not divide $x_3^{a_3} x_4^{a_4}$.

From (7), one has

$$\begin{aligned} f + n_1 =& (\alpha_2 + \alpha_{32} - 1) n_2 + (\alpha_{13} - 1) n_3 + (\alpha_4 - 1) n_4 \\ =& (\alpha_2 - 1) n_2 + (\alpha_{13} + \alpha_3 - 1) n_3 + (\alpha_{14} - 1) n_4. \end{aligned}$$

These are two decompositions of $f+n_1$ with respect to G and have the same length. It will be routine to check that $x \preceq_G (f+n_1)$.

(xi) $(i,j,h,k)=(3,4,1,2)$. Then

$$I=(f_1:=x_3^{\alpha_3}-x_4^{\alpha_4}, f_2:=x_1^{\alpha_1}-x_3^{\alpha_{13}}x_4^{\alpha_{14}}, f_3:=x_2^{\alpha_2}-x_1^{\alpha_{21}}x_3^{\alpha_{23}}x_4^{\alpha_{24}}).$$

Obviously $f_1^*=-x_4^{\alpha_4}$ and $f_2^*=-x_3^{\alpha_{13}}x_4^{\alpha_{14}}$. We can assume from the beginning that $0\le \alpha_{14},\alpha_{24}<\alpha_4$, and $x_3^{\alpha_{23}}x_4^{\alpha_{24}}$ is not divisible by $x_3^{\alpha_{13}}x_4^{\alpha_{14}}$.

(A) If $\alpha_{14}=0$, then $\alpha_{13}\ge\alpha_3$. We have two sub-cases.

(a) If $\alpha_{21}\neq 0$ or $\alpha_{24}\neq 0$, then f_1^*, f_2^* and f_3^* form part of a minimal system of generators for I^*. Since f_2^* is a pure power in x_3, I^* has to be a complete intersection after modulo x_1. This can happen only when $\alpha_2\le\alpha_{21}+\alpha_{23}+\alpha_{24}$. The rest of the proof for this case is similar to that of (i).

(b) If $\alpha_{21}=\alpha_{24}=0$, then $\alpha_3\le\alpha_{23}<\alpha_{13}$. One can use f_3 to reduce the problem to the case (xii) with $\alpha_{12}\neq 0$.

(B) If $\alpha_{14}\neq 0$, since $\alpha_{14}<\alpha_4$, one must have $\alpha_{13}>0$. Now consider the following two sub-cases.

(a) If $\alpha_2\le\alpha_{21}+\alpha_{23}+\alpha_{24}$, then the initial monomial of f_3 is $x_2^{\alpha_2}$. Now apply the standard basis algorithm. We get

$$f_4:=\mathrm{spoly}(f_1,f_2)=x_3^{\alpha_3+\alpha_{13}}-x_1^{\alpha_1}x_4^{\alpha_4-\alpha_{14}}.$$

None of the two monomial terms is divisible by any of f_1^*, f_2^* or f_3^*. Hence f_4 is one new generator for the standard basis. Since $\alpha_1>0$, by the Cohen-Macaulay condition, we need to have $\alpha_3+\alpha_{13}\le\alpha_1+\alpha_4-\alpha_{14}$. Now $\mathrm{spoly}(f_2,f_4)=x_1^{\alpha_1}f_1$. Hence the standard basis algorithm stops at this step, and the standard basis has only 4 elements. In other words, the initial form ideal I^* is an almost complete intersection, and cannot be Gorenstein. One can also use the structure property for strictly almost monomial ideals, by noticing that it should not contain pure powers in x_3.

(b) If $\alpha_2>\alpha_{21}+\alpha_{23}+\alpha_{24}$, then $f_3^*=-x_1^{\alpha_{21}}x_3^{\alpha_{23}}x_4^{\alpha_{24}}$.
If f_1^*, f_2^* and f_3^* form part of the minimal system of generators, then by the Cohen-Macaulay condition, we must have $\alpha_{21}=0$. Since f_2^* is not a pure power, I^* need to be a strictly almost monomial ideal after modulo x_1. If f_3^* is a pure power in x_3, then I^* need to be a complete intersection. This is impossible, since $\alpha_{14}\neq 0$. If $\alpha_{24}\neq 0$, then $\alpha_{23}>0$. But in an almost monomial ideal of our kind, monomials in both x_3 and x_4 only show up once. Hence we have another contradiction.
If f_1^*, f_2^* and f_3^* do not form part of the minimal system of generators, since by our assumption $\alpha_{24}<\alpha_4$ and $x_3^{\alpha_{23}}x_4^{\alpha_{24}}$ is not divisible by $x_3^{\alpha_{13}}x_4^{\alpha_{14}}$, this can only happen when f_2^* is divisible by f_3^*. In particular, $\alpha_{21}=0$. Again, we can use f_3 to reduce the problem to the case (xii) with $\alpha_{12}\neq 0$.

(xii) $(i,j,h,k)=(3,4,2,1)$. Then

$$I=(f_1:=x_3^{\alpha_3}-x_4^{\alpha_4}, f_2:=x_2^{\alpha_2}-x_3^{\alpha_{23}}x_4^{\alpha_{24}}, f_3:=x_1^{\alpha_1}-x_2^{\alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}}).$$

This case is similar to that of (xi). $f_1^* = -x_4^{\alpha_4}$, $f_2^* = -x_3^{\alpha_{23}}x_4^{\alpha_{24}}$ and $f_3^* = -x_2^{\alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}}$. We can assume that $0 \leq \alpha_{14}, \alpha_{24} < \alpha_4$ and $x_3^{\alpha_{13}}x_4^{\alpha_{14}}$ is not divisible by $x_3^{\alpha_{23}}x_4^{\alpha_{24}}$.

(A) Suppose f_1^*, f_2^* and f_3^* form part of a minimal system of generators for I^*. Since I^* is almost monomial after modulo x_1, we have the following cases to consider with all given exponents strictly positive.

(a) $f_3^* = -x_2^{\alpha_{12}}$ with $\alpha_{12} \geq \alpha_2$. Applying standard basis algorithm, we get $f_4 = \mathrm{spoly}(f_1, f_2) = x_2^{\alpha_2}x_4^{\alpha_4-\alpha_{24}} - x_3^{\alpha_3+\alpha_{23}}$ and it should be a new generator for the standard basis. In almost monomial ideal described in 5.7, monomial of the form $x_2^{\alpha_2}x_4^{\alpha_4-\alpha_{24}}$ is not a minimal generator. Hence if I^* is Gorenstein, we must have $\alpha_3 + \alpha_{23} \leq \alpha_2 + \alpha_4 - \alpha_{24}$. Whence the initial monomial of f_4 is $-x_3^{\alpha_3+\alpha_{23}}$. Now the algorithm stops at this step and $I^* = (f_1^*, f_2^*, f_3^*, f_4^*)$ is an almost complete intersection, hence not Gorenstein.

(b) $f_3^* = -x_3^{\alpha_{13}}$ with $\alpha_{13} \geq \alpha_3$. Since it is a pure power in x_3, the initial form ideal I^* has to be a complete intersection after modulo x_1. However, f_2^* is also an irredundant generator. This gives a contradiction.

(c) $f_3^* = -x_3^{\alpha_{13}}x_4^{\alpha_{14}}$. Since f_3^* is irredundant, f_2^* cannot have strictly positive exponents for both variables. But $0 \leq \alpha_{24} < \alpha_4$, hence $\alpha_{23} \neq 0$. Thus $\alpha_{24} = 0$ and f_2^* is a pure power in x_3. As a result, I^* need to be an complete intersection after modulo x_1. This is impossible, because of f_3^*.

(d) $f_3^* = -x_2^{\alpha_{12}}x_3^{\alpha_{13}}$. By the structure of almost monomial ideal discussed in 5.7, we must have $\alpha_{24} \neq 0$ and $\alpha_{13} = \alpha_{23}$. Apply the standard basis algorithm, we get two more expected generators

$$f_4 := \mathrm{spoly}(f_1, f_2) = x_2^{\alpha_2}x_4^{\alpha_4-\alpha_{24}} - x_3^{\alpha_3+\alpha_{23}}$$

and

$$f_5 := \mathrm{spoly}(f_2, f_3) = x_2^{\alpha_2+\alpha_{12}} - x_1^{\alpha_1}x_4^{\alpha_{24}}.$$

Meanwhile, we get two necessary conditions $\alpha_3 + \alpha_{23} = \alpha_2 + \alpha_4 - \alpha_{24}$ and $\alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_{24}$. By (7), we know

$$\begin{aligned} f + n_1 =&(\alpha_2 + \alpha_{12} - 1)n_2 + (\alpha_{13} - 1)n_3 + (\alpha_4 - 1)n_4 \\ =&(\alpha_{12} - 1)n_2 + (2\alpha_{13} + \alpha_3 - 1)n_3 + (\alpha_{24} - 1)n_4 \end{aligned}$$

The rest of the proof is similar to that of (c) in (x).

(B) If f_1^*, f_2^* and f_3^* fail to be part of a system of minimal generators, then f_3^* must divide f_2^*. This means $\alpha_{12} = 0$ as well. Now using f_3, the problem can be reduced to the case (xi) with $\alpha_{21} \neq 0$.

5.3. **Proof for case III.** Up to a permutation of variables, the defining ideal I has the following form:

$$\begin{gathered} I = (x_1^{\alpha_1} - x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}}, \\ x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}, x_4^{\alpha_4} - x_2^{\alpha_{42}}x_3^{\alpha_{43}}, x_3^{\alpha_{43}}x_1^{\alpha_{21}} - x_2^{\alpha_{32}}x_4^{\alpha_{14}}). \end{gathered}$$

In this case, Bresinsky [3, Theorem 5] proved that $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$ and $\alpha_4 = \alpha_{14} + \alpha_{24}$.

To simplify the notation, we follow the convention in [2] and write it as

$$f_1 = (1,(3,4)), f_2 = (2,(1,4)), f_3 = (3,(1,2)),$$
$$f_4 = (4,(2,3)), f_5 = ((1,3),(2,4)).$$

Hence we have six cases to consider

1. $f_1 = (1,(3,4))$
   (a) $f_2 = (2,(1,4))$, $f_3 = (3,(1,2))$, $f_4 = (4,(2,3))$, $f_5 = ((1,3),(2,4))$
   (b) $f_2 = (2,(1,3))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,2))$, $f_5 = ((1,4),(2,3))$
2. $f_1 = (1,(2,3))$
   (a) $f_2 = (2,(3,4))$, $f_3 = (3,(1,4))$, $f_4 = (4,(1,2))$, $f_5 = ((2,4),(1,3))$
   (b) $f_2 = (2,(1,4))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,3))$, $f_5 = ((1,2),(4,3))$
3. $f_1 = (1,(2,4))$
   (a) $f_2 = (2,(1,3))$, $f_3 = (3,(1,4))$, $f_4 = (4,(2,3))$, $f_5 = ((1,2),(3,4))$
   (b) $f_2 = (2,(3,4))$, $f_3 = (3,(1,2))$, $f_4 = (4,(1,3))$, $f_5 = ((2,3),(1,4))$

We also need the following result.

Theorem 5.12 ([2, 2.10])**.** *In case III of Theorem 5.2, write* $f_1 = x_1^{\alpha_1} - m_1$, $f_2 = x_2^{\alpha_2} - m_2$, $f_3 = x_3^{\alpha_3} - m_3$, $f_4 = x_4^{\alpha_4} - m_4$ *and* f_5*, where* m_1, m_2, m_3 *and* m_4 *are monomials. If* $\alpha_2 \leq$ *total degree of* m_2 *and* $\alpha_3 \leq$ *total degree of* m_3*, then the tangent cone* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay.*

Their conditions are equivalent to saying:

case 1(a): $\alpha_2 \leq \alpha_{21} + \alpha_{24}$;
case 1(b): $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$;
case 2(b): $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$;
case 3(a): $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{31} + \alpha_{34}$.

No similar result is reproduced for 2(a) and 3(b), since $f_2 = x_2^{\alpha_2} - x_3^{\alpha_{23}} x_4^{\alpha_{24}}$ is an element of the generator set in both cases, and thus $\alpha_2 > \alpha_{23} + \alpha_{24}$.

It follows from 3.2 that if $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then I^* does not contain any monomial minimal generator that is divisible by x_1.

Corollary 5.13. *Assume the setting of 5.2, if the defining ideal I is not a complete intersection and the associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay, then the following inequalities hold:*

case 1(a): $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_3 \leq \alpha_{31} + \alpha_{32}$, $\alpha_{21} + \alpha_{43} \geq \alpha_{32} + \alpha_{14}$;
case 1(b): $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_4 \leq \alpha_{41} + \alpha_{42}$, $\alpha_{21} + \alpha_{34} \geq \alpha_{42} + \alpha_{13}$;
case 2(a): $\alpha_3 \leq \alpha_{31} + \alpha_{34}$, $\alpha_4 \leq \alpha_{41} + \alpha_{42}$, $\alpha_{41} + \alpha_{23} \geq \alpha_{12} + \alpha_{34}$;
case 2(b): $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_4 \leq \alpha_{41} + \alpha_{43}$, $\alpha_{41} + \alpha_{32} \geq \alpha_{24} + \alpha_{13}$;
case 3(a): $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{31} + \alpha_{34}$, $\alpha_{31} + \alpha_{42} \geq \alpha_{23} + \alpha_{14}$;
case 3(b): $\alpha_3 \leq \alpha_{31} + \alpha_{32}$, $\alpha_4 \leq \alpha_{41} + \alpha_{43}$, $\alpha_{41} + \alpha_{34} \geq \alpha_{31} + \alpha_{24}$.

Some of the inequalities are redundant, and hold automatically even without assuming the Cohen-Macaulayness of $\mathrm{gr}_{\mathfrak{m}}(R)$. They are kept here for completeness. But as an interesting result, we can see that

Corollary 5.14. *For cases 1(a) and 3(a), the conditions in* [2, 2.10] *are necessary and sufficient for* $\mathrm{gr}_{\mathfrak{m}}(R)$ *to be Cohen-Macaulay.*

Due to 2.5, our proof for 5.1 will be done by showing the following.

Proposition 5.15. *Assume the notations in 5.12. Then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein if and only the following conditions hold:*

(a) $\alpha_2 \leq$ *total degree of* m_2,
(b) $f_3 = (3,(2,4))$ *and* $\alpha_3 =$ *total degree of* m_3.

In particular, $\mathrm{gr}_{\mathfrak{m}}(R)$ *could be Gorenstein only for case 1(b) and case 2(b).*

Proof. Sufficiency. Condition (2) only holds for cases 1(b) or 2(b). According to [2, 2.10], $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay and the initial form ideal $I^* \subset S = k[x_1, x_2, x_3, x_4]$ of I is generated by the initial form of each generators.

In case 1(b) we have

$$I^* = (x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}} \text{ or } x_2^{\alpha_2}, x_3^{\alpha_3} - x_2^{\alpha_{32}}x_4^{\alpha_{34}}, x_4^{\alpha_4}, (x_1^{\alpha_{21}}x_4^{\alpha_{34}} - x_2^{\alpha_{42}}x_3^{\alpha_{13}})^*).$$

We want to point out that $\alpha_{21} + \alpha_{34} \geq \alpha_{42} + \alpha_{13}$. To see this inequality, just note that by the assumptions we have

$$\alpha_2 = \alpha_{32} + \alpha_{42} \leq \alpha_{21} + \alpha_{23} \quad \text{and} \quad \alpha_3 = \alpha_{13} + \alpha_{23} = \alpha_{32} + \alpha_{34}.$$

Now it follows that

$$(x_1^{\alpha_{21}}x_4^{\alpha_{34}} - x_2^{\alpha_{42}}x_3^{\alpha_{13}})^* = (x_1^{\alpha_{21}}x_4^{\alpha_{34}} - x_2^{\alpha_{42}}x_3^{\alpha_{13}}) \text{ or } -x_2^{\alpha_{42}}x_3^{\alpha_{13}}.$$

Hence after modulo x_1, I^* becomes

$$\widetilde{I^*} = (x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3} - x_2^{\alpha_{32}}x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_2^{\alpha_{42}}x_3^{\alpha_{13}}).$$

It follows from 5.5 that I^* is a Gorenstein ideal.

As for case 2(b), we have $f_5 = x_1^{\alpha_{41}}x_2^{\alpha_{32}} - x_3^{\alpha_{13}}x_4^{\alpha_{24}}$. Hence $\alpha_{41}n_1 + \alpha_{32}n_2 = \alpha_{13}n_3 + \alpha_{24}n_4$. But $\alpha_{41}n_1 + \alpha_{32}n_2 < \alpha_{41}n_2 + \alpha_{32}n_2$ and $\alpha_{13}n_3 + \alpha_{24}n_4 > \alpha_{13}n_2 + \alpha_{24}n_2$. This implies that $\alpha_{41} + \alpha_{32} > \alpha_{13} + \alpha_{24}$, and $f_5^* = -x_3^{\alpha_{13}}x_4^{\alpha_{24}}$. The rest of the discussion is similar to that of the previous case.

Necessity. Since $0 < \alpha_{ij} < \alpha_j$, the five generators f_i form part of a minimal standard basis for I, and their initial forms f_i^* form part of the minimal basis for the leading ideal I^*. By the Cohen-Macaulay assumption, after modulo x_1, none of the leading forms will go to zero. Hence $\widetilde{f_i^*}$ form part of the minimal basis for $\widetilde{I^*} \subset k[x_2, x_3, x_4]$, where $\tilde{}$ denotes the image $k[x_1, x_2, x_3, x_4]/(x_1) \cong k[x_2, x_3, x_4]$. By the Gorenstein assumption, this $\widetilde{I^*}$ is again Gorenstein, and not a complete intersection. Hence by 5.7 it is strictly almost monomial and 5-generated. Now the assertion follows from 5.6 and 5.13. $\square$

Corollary 5.16. *Assume that the numerical semigroup G is 4-generated, and the defining ideal I is not a complete intersection. If the associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein, then the elasticity of the special element $f + n_1$ is 1 and* $\mathrm{ord}_G(f + n_1) = \alpha_2 + \alpha_4 + \alpha_{13} - 3$.

Proof. By 5.15, we only need to consider the two cases 1(b) and 2(b). We can write $f + n_1 = z_1n_1 + z_2n_2 + z_3n_3 + z_4n_4$. Since $t^{f+n_1} \notin (t^{n_1})$, we must have $z_1 = 0$, and hence $z_2 \leq \alpha_2 - 1$ and $z_4 \leq \alpha_4 - 1$. Suppose that we can also write $f + n_1 = z_2'n_2 + z_3'n_3 + z_4'n_4$. By symmetry, there are three cases to consider.

(a) $z_2' > z_2$, $z_3' \leq z_3$ and $z_4' \leq z_4$.
(b) $z_3' > z_3$, $z_2' \leq z_2$ and $z_4' \leq z_4$.
(c) $z_4' > z_4$, $z_2' \leq z_2$ and $z_3' \leq z_3$.

By the choice of α_2, α_3 and α_4, the first and third case cannot happen. Because otherwise, for instance in the first case, we would have $(z_2' - z_2)n_2 = (z_3 - z_3')n_3 + (z_4 - z_4')n_4$. Since $z_2' - z_2 < \alpha_2$, this cannot happen.

By the same reason, in the second case, one must have $z_2' = z_2 - k\alpha_{32}$, $z_3' = z_3 + k\alpha_3$ and $z_4' = z_4 - k\alpha_{34}$ for some $k \in \mathbb{N}_0$. Since $\alpha_3 = \alpha_{32} + \alpha_{34}$, we can conclude that $\sum z_i = \sum z_i'$, i.e., the elasticity of $f + n_1$ is 1.

Now by the proof of [3, Theorem 5], one knows that $f + n_1 = (\alpha_2 - 1)n_2 + (\alpha_{13} - 1)n_3 + (\alpha_4 - 1)n_4$. It is clear then $(\alpha_2 - 1) + (\alpha_{13} - 1) + (\alpha_4 - 1)$ is one (and hence the) length of $f + n_1$. □

Corollary 5.17. *In case III of 5.2, if* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Gorenstein, then the initial form ideal* I^* *is 5-generated.*

Remark 5.18. The converse of 5.17 is not true. For instance in case 1(a) of 5.14, when $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, I^* is 5-generated. But it can never be Gorenstein by 5.15.

Acknowledgement

The author wishes to thank Bernd Ulrich for the helpful suggestions throughout the preparation of this paper. Also, the author wishes to thank William Heinzer for the warm encouragement and helpful comments. He wants to acknowledge the support provided by Singular [9] and GAP [7]. In particular, he thanks Lance Bryant for the many helpful conversations and for the Singular library that facilitates the computations of initial form ideals.

Appendix

Here we use the standard basis method to give a strict proof of formula (2). Without loss of generality, we can assume that $\alpha < a$, $\beta < b - b'$ and $\gamma < c - c'$.

Consider the following monomials and binomials in $k[t, x, y, z]$ with lexicographical monomial ordering.

$$
\begin{aligned}
f_1 &= tx^a - ty^{b'}z^{c'} & f_2 &= ty^b \\
f_3 &= tz^c & f_4 &= tx^\alpha y^\beta z^\gamma - x^\alpha y^\beta z^\gamma \\
f_5 &= ty^{b'+\beta}z^{c'+\gamma} - x^a y^\beta z^\gamma & f_6 &= x^\alpha y^b z^\gamma \\
f_7 &= x^\alpha y^\beta z^c & f_8 &= x^a y^{b-b'} z^\gamma \\
f_9 &= x^a y^\beta z^{c-c'} & f_{10} &= x^{\alpha+a} y^\beta z^\gamma - x^\alpha y^{\beta+b'} z^{\gamma+c'}
\end{aligned}
$$

The following table shows that every s-polynomial $\mathrm{spoly}(f_i, f_j)$ is either 0 or can be reduced by some f_k. By symmetry, we only need to consider for $i < j$.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
f_1		f_2	f_3	f_5	f_5, f_8	f_2	f_3	f_2	f_3	0
f_2			0	f_6	f_8	0	0	0	0	f_2
f_3				f_7	f_9	0	0	0	0	f_3
f_4					f_{10}	f_6	f_7	f_8	f_9	f_5
f_5						f_8	f_9	f_{10}, f_8	f_{10}, f_9	f_{10}
f_6							0	0	0	f_6
f_7								0	0	f_7
f_8									0	f_6
f_9										f_7
f_{10}										

Since $f_5 = \mathrm{spoly}(f_1, f_4)$, $f_6 = \mathrm{spoly}(f_2, f_4)$, $f_7 = \mathrm{spoly}(f_3, f_4)$, $f_8 = \mathrm{spoly}(f_2, f_5)$, $f_9 = \mathrm{spoly}(f_3, f_5)$ and $f_{10} = \mathrm{spoly}(f_4, f_5)$, we know immediately that the set $\{f_i : 1 \le i \le 10\}$ forms a standard basis for $I = (f_1, f_2, f_3, f_4)$. Hence by [8, 1.8.10], the intersection ideal is

$$\begin{aligned}(x^a - y^{b'}z^{c'}, y^b, z^c) \cap (x^\alpha y^\beta z^\gamma) = (&x^\alpha y^\beta z^\gamma \cdot y^{b-\beta}, x^\alpha y^\beta z^\gamma \cdot z^{c-\gamma},\\ &x^\alpha y^\beta z^\gamma \cdot x^{a-\alpha} y^{b-b'-\beta}, x^\alpha y^\beta z^\gamma \cdot x^{a-\alpha} z^{c-c'-\gamma}, x^\alpha y^\beta z^\gamma \cdot (x^a - y^{b'}z^{c'}))\end{aligned}$$

and by [8, 1.8.12], the quotient ideal is

$$(x^a - y^{b'}z^{c'}, y^b, z^c) : x^\alpha y^\beta z^\gamma = (x^a - y^{b'}z^{c'}, y^{b-\beta}, z^{c-\gamma}, x^{a-\alpha} y^{b-b'-\beta}, x^{a-\alpha} z^{c-c'-\gamma}).$$

References

[1] F. Arslan, Cohen-Macaulayness of tangent cones, *Proc. Amer. Math. Soc.* **128** (2000) 2243–2251.

[2] F. Arslan, P. Mete, Hilbert functions of Gorenstein monomial curves, *Proc. Amer. Math. Soc.* **135** (2007) 1993–2002.

[3] H. Bresinsky, Symmetric semigroups of integers generated by 4 elements, *Manuscripta Math.* **17** (1975) 205–219.

[4] H. Bresinsky, Monomial Gorenstein ideals, *Manuscripta Math.* **29** (1979) 159–181.

[5] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, vol. 39 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1993.

[6] L. Bryant, The goto numbers of a numerical semigroup ring and the gorensteinness of associated graded rings, preprint.

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.10, `http://www.gap-system.org` (2007).

[8] G.-M. Greuel, G. Pfister, *A Singular introduction to commutative algebra*, Springer-Verlag, Berlin, 2002.

[9] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). `http://www.singular.uni-kl.de`.

[10] W. Heinzer, I. Swanson, The Goto numbers of parameter ideals, preprint, `arXiv:0801.0685v1 [math.AC]`.

[11] J. Herzog, Generators and relations of abelian semigroups and semigroup rings., *Manuscripta Math.* **3** (1970) 175–193.

[12] J. Herzog, When is a regular sequence super regular?, *Nagoya Math. J.* **83** (1981) 183–195.

[13] C. Huneke, B. Ulrich, Liaison of monomial ideals, *Bull. Lond. Math. Soc.* **39** (2007) 384–392.

[14] L. Robbiano, G. Valla, On the equations defining tangent cones, *Math. Proc. Cambridge Philos. Soc.* **88** (1980) 281–297.

[15] V. A. Sapko, Associated graded rings of numerical semigroup rings, *Comm. Algebra* **29** (2001) 4759–4773.

[16] T. Shibuta, Cohen-macaulayness of almost complete intersection tangent cones, *J. Algebra* **319** (2008) 3222–3243.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, U.S.A.

E-mail address: `yshen@math.purdue.edu`